

Hyperkähler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace

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ABSTRACT: Kähler manifolds have a natural hyperkähler structure associated with (part of) their cotangent bundles. Using projective superspace, we construct four-dimensional $\mathcal{N} = 2$ models on the tangent bundles of some classical Hermitian symmetric spaces (specifically, the four regular series of irreducible compact symmetric Kähler manifolds, and their non-compact versions). A further dualization yields the Kähler potential for the hyperkähler metric on the cotangent bundle.

KEYWORDS: Sigma Models, Extended Supersymmetry, Differential and Algebraic Geometry.

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1. Introduction

Supersymmetry in sigma models is closely related to the geometry of target space [1]. In particular, $\mathcal{N} = 2$ models in four space-time dimensions require the target space geometry to be hyperkähler [2]. Consequently, constructions of new $\mathcal{N} = 2$ sigma models lead to new hyperkähler metrics, a fact which has been extensively pursued in the Legendre transform and hyperkähler quotient constructions [3, 4].

To fully utilize the relation to geometry, manifest $\mathcal{N} = 2$ formulations are needed. Projective superspace [5, 4] provides this, and has led to the discovery of a number of new

multiplets that can be used to construct new hyperkähler metrics (see e.g. [6]). In [6] a generalized Legendre transform was devised, that produces hyperkähler metrics.¹

Among the projective supermultiplets, perhaps the most interesting one is the so-called polar multiplet [6] which can be used to describe a charged $U(1)$ hypermultiplet coupled to a vector multiplet [7], and therefore is analogous to the $\mathcal{N} = 1$ chiral superfield. The polar multiplet² is described by an arctic superfield $\Upsilon(\zeta)$ and its complex conjugate composed with the antipodal map $\bar{\zeta} \rightarrow -1/\zeta$, the antarctic superfield $\check{\Upsilon}(\zeta)$. It is required to possess certain holomorphy properties on a punctured two-plane parametrized by the complex variable ζ (the latter may be interpreted as a projective coordinate on $\mathbb{C}P^1$). When realized in ordinary $\mathcal{N} = 1$ superspace, $\Upsilon(\zeta)$ and $\check{\Upsilon}(\zeta)$ are generated by an infinite set of ordinary superfields:

$$\Upsilon(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma \zeta + O(\zeta^2), \quad \check{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} \check{\Upsilon}_n (-\zeta)^{-n}. \quad (1.1)$$

Here Φ is chiral, Σ complex linear,

$$\bar{D}_{\dot{\alpha}} \Phi = 0, \quad \bar{D}^2 \Sigma = 0, \quad (1.2)$$

and the remaining component superfields are unconstrained complex superfields. Using the polar multiplet, one can construct a family of 4D $\mathcal{N} = 2$ off-shell supersymmetric nonlinear sigma-models that are described in $\mathcal{N} = 1$ superspace by the action

$$S[\Upsilon, \check{\Upsilon}] = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^8z K(\Upsilon(\zeta), \check{\Upsilon}(\zeta), \zeta), \quad (1.3)$$

with the integration contour around the origin in \mathbb{C} .

The unconstrained superfields $\Upsilon_2, \Upsilon_3, \dots$, and their conjugates, appear in the action (1.3) without derivatives, and therefore they are purely auxiliary degrees of freedom. Their role is to ensure a linearly realized $\mathcal{N} = 2$ supersymmetry. In order to describe the theory only in terms of the physical superfields $\Upsilon_0 = \Phi$ and $\Upsilon_1 = \Sigma$, one has to eliminate all the auxiliary superfields using their equations of motion. The problem of elimination of the auxiliary superfields is actually nontrivial, since one has to solve an infinite number of nonlinear equations. So far it has been solved perturbatively only for a broad subclass of the models (1.3) studied in [9–11], and also exact solutions have been found in special cases [10–13]. Such a family is obtained by restricting $K(\Upsilon, \check{\Upsilon}, \zeta) \rightarrow K(\Upsilon, \check{\Upsilon})$ in (1.3). Then, the corresponding action can be viewed as a minimal $\mathcal{N} = 2$ extension of the general four-dimensional $\mathcal{N} = 1$ supersymmetric nonlinear sigma model [1], with $K(\Phi, \bar{\Phi})$ the Kähler potential of a Kähler manifold \mathcal{M} , and the physical superfields (Φ, Σ) parametrizing the tangent bundle $T\mathcal{M}$ of the Kähler manifold [9]. Upon elimination of the auxiliary superfields, the complex linear tangent variables Σ can be dualized (as a final step of the generalized Legendre transform [6]) into chiral one-forms, such that the target space for the model obtained turns out to be (an open domain of the zero section of) the cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold [10, 11].

¹The term “projective superspace” was coined in [7].

²The terminology “polar” and “(ant)arctic” multiplets was introduced in [8].

The perturbative procedure for the elimination of the auxiliary superfields, which was originally described in [10], has recently been refined by one of us (SMK), see e.g. [14]. The scheme obtained is reminiscent of the mathematical techniques used to prove the theorem [15] that, for a Kähler manifold \mathcal{M} , a canonical hyperkähler structure exists, in general, on an open neighborhood of the zero section of the cotangent bundle $T^*\mathcal{M}$.

As outlined in [10] and further elaborated in [13], for Hermitian symmetric spaces the auxiliary fields may be eliminated exactly. In the present paper we make systematic use of this fact. The method entails finding a particular solution to the auxiliary field equations at the origin of \mathcal{M} (in a normal coordinate system³ introduced in [16, 17]) and then relying on the existence of holomorphic isometries and other special properties to extend the solution to an arbitrary point. A key ingredient in this procedure is a coset representative of a convenient form.

Let us end the introduction with a brief comment on related work. The first hyperkähler manifolds that are cotangent bundles of certain complex Grassmannians (generalizations of the Calabi manifolds), were presented in [3]. Lately, massive versions of these and related models have been discussed in [19–22] using $\mathcal{N} = 1$ superspace and $\mathcal{N} = 2$ harmonic superspace techniques [23]. In addition to these specific examples, some structural results for massive $\mathcal{N} = 2$ sigma models have been obtained in $\mathcal{N} = 1$ superspace [24] and projective superspace [25].

The presentation of the paper is organized as follows. Section 2 describes the type of $\mathcal{N} = 2$ supersymmetric sigma models we are interested in and their background in projective superspace. In section 3 we construct the coset representatives needed for the four types of compact Hermitian symmetric spaces. Section 4 contains the construction of the sigma models for a Grassmann manifold and is followed in sections 5 and 6 by the construction for the remaining three symmetric spaces. In section 7 we repeat the discussions in section 3, but now for the non-compact versions of the spaces. The corresponding sigma models occupy sections 8, 9 and 10. In two appendices we present derivations of results needed in the general text.

The results obtained in the present paper admit a natural extension to 5D [14] and 6D [26, 27] projective superspace formulations.

2. Dynamical setup

This section sets the stage for our constructions of supersymmetric sigma models on symmetric spaces. In particular we introduce the relevant $\mathcal{N} = 2$ extensions of $\mathcal{N} = 1$ sigma models and their origin in projective superspace.

Projective superspace is a superspace enlarged with a $\mathbb{C}P^1$ at each point. The coordinates thus include an additional projective coordinate ζ on this space. The larger space allows for integrations over invariant subspaces, much like the chiral integrals in ordinary superspace, and hence for manifest $\mathcal{N} = 2$ actions. The integration measure contains a contour integral over a closed curve in the complex ζ -plane which picks out the residue

³For such coordinates, the term “Kähler normal coordinates” was suggested in [18].

from the Lagrangian. Projective superspace was introduced in [5, 4], and has been continuously developed over the years. For a recent mathematically oriented description which also elaborates the close relation to twistor space, see [28].

In this paper, we are interested in a family of 4D $\mathcal{N} = 2$ off-shell supersymmetric nonlinear sigma-models that are described in ordinary $\mathcal{N} = 1$ superspace by the action

$$S[\Upsilon, \check{\Upsilon}] = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^8z K(\Upsilon^I(\zeta), \check{\Upsilon}^{\bar{I}}(\zeta)) . \quad (2.1)$$

These dynamical systems present themselves a subclass of the more general family of 4D $\mathcal{N} = 2$ off-shell supersymmetric nonlinear models [6] in projective superspace, given in eq. (1.3). What is special about the model (2.1) is its interesting geometric properties [9–11]. It occurs as a minimal $\mathcal{N} = 2$ extension of the general four-dimensional $\mathcal{N} = 1$ supersymmetric nonlinear sigma model [1]

$$S[\Phi, \bar{\Phi}] = \int d^8z K(\Phi^I, \bar{\Phi}^{\bar{J}}), \quad (2.2)$$

with K the Kähler potential of a Kähler manifold \mathcal{M} .

The extended supersymmetric sigma model (2.1) inherits all the geometric features of its $\mathcal{N} = 1$ predecessor (2.2). The Kähler invariance of the latter,

$$K(\Phi, \bar{\Phi}) \longrightarrow K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \quad (2.3)$$

turns into

$$K(\Upsilon, \check{\Upsilon}) \longrightarrow K(\Upsilon, \check{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\check{\Upsilon}) \quad (2.4)$$

for the model (2.1). A holomorphic reparametrization of the Kähler manifold,

$$\Phi^I \longrightarrow f^I(\Phi), \quad (2.5)$$

has the following counterpart

$$\Upsilon^I(\zeta) \longrightarrow f^I(\Upsilon(\zeta)) \quad (2.6)$$

in the $\mathcal{N} = 2$ case. Therefore, the physical superfields of the $\mathcal{N} = 2$ theory

$$\Upsilon^I(\zeta) \Big|_{\zeta=0} = \Phi^I, \quad \frac{d\Upsilon^I(\zeta)}{d\zeta} \Big|_{\zeta=0} = \Sigma^I, \quad (2.7)$$

should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point. Thus the variables (Φ^I, Σ^J) parametrize the tangent bundle $T\mathcal{M}$ of the Kähler manifold \mathcal{M} [9].

To describe the theory in terms of the physical superfields Φ and Σ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{d\zeta}{\zeta} \zeta^n \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{d\zeta}{\zeta} \zeta^{-n} \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \check{\Upsilon}^{\bar{I}}} = 0, \quad n \geq 2 . \quad (2.8)$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ denote a unique solution subject to the initial conditions

$$\Upsilon_*(0) = \Phi, \quad \dot{\Upsilon}_*(0) = \Sigma. \quad (2.9)$$

For a general Kähler manifold \mathcal{M} , the auxiliary superfields $\Upsilon_2, \Upsilon_3, \dots$, and their conjugates, can be eliminated only perturbatively. Their elimination can be carried out using the ansatz [14]

$$\Upsilon_n^I = \sum_{p=0}^{\infty} G^I{}_{J_1 \dots J_{n+p} \bar{L}_1 \dots \bar{L}_p}(\Phi, \bar{\Phi}) \Sigma^{J_1} \dots \Sigma^{J_{n+p}} \bar{\Sigma}^{\bar{L}_1} \dots \bar{\Sigma}^{\bar{L}_p}, \quad n \geq 2. \quad (2.10)$$

Upon elimination of the auxiliary superfields, the action (2.1) takes the form [10, 11]

$$S_{\text{tb}}[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}] = \int d^8z \left\{ K(\Phi, \bar{\Phi}) - g_{I\bar{J}}(\Phi, \bar{\Phi}) \Sigma^I \bar{\Sigma}^{\bar{J}} + \sum_{p=2}^{\infty} \mathcal{R}_{I_1 \dots I_p \bar{J}_1 \dots \bar{J}_p}(\Phi, \bar{\Phi}) \Sigma^{I_1} \dots \Sigma^{I_p} \bar{\Sigma}^{\bar{J}_1} \dots \bar{\Sigma}^{\bar{J}_p} \right\}, \quad (2.11)$$

where the tensors $\mathcal{R}_{I_1 \dots I_p \bar{J}_1 \dots \bar{J}_p}$ are functions of the Riemann curvature $R_{I\bar{J}K\bar{L}}(\Phi, \bar{\Phi})$ and its covariant derivatives. Each term in the action contains equal powers of Σ and $\bar{\Sigma}$, since the original model (2.1) is invariant under rigid U(1) transformations [10]

$$\Upsilon(\zeta) \mapsto \Upsilon(e^{i\alpha} \zeta) \iff \Upsilon_n(z) \mapsto e^{in\alpha} \Upsilon_n(z). \quad (2.12)$$

The process of eliminating the auxiliary fields from the action (2.1) and subsequently performing a Legendre transform with respect to linear fields is called a generalized Legendre transform [6]. For the theory with action $S_{\text{tb}}[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}]$, this gives a dual formulation involving only chiral superfields and their conjugates as the dynamical variables. Consider the first-order action

$$S_{\text{tb}}[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}] + \int d^8z \left\{ \psi_I \Sigma^I + \bar{\psi}_{\bar{I}} \bar{\Sigma}^{\bar{I}} \right\}, \quad (2.13)$$

where the tangent vector Σ^I is now complex unconstrained, while the one-form ψ_I is chiral, $\bar{D}_{\dot{\alpha}} \psi_I = 0$. Upon elimination of Γ and $\bar{\Gamma}$, with the aid of their equations of motion, the action turns into

$$S_{\text{cb}}[\Phi, \bar{\Phi}, \psi, \bar{\psi}] = \int d^8z H(\Phi, \bar{\Phi}, \psi, \bar{\psi}). \quad (2.14)$$

Its target space is (an open domain of the zero section of) the cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold \mathcal{M} , and $H(\Phi, \bar{\Phi}, \psi, \bar{\psi})$ the corresponding hyperkähler potential [10, 11].

For Hermitian symmetric spaces, the auxiliary superfields can in principle be eliminated exactly, as outlined in [10]. Here we present a more elaborated procedure following mainly [13].

Given a Kähler manifold \mathcal{M} , and an arbitrary point $p_0 \in \mathcal{M}$, one can construct a Kähler normal coordinate system with the origin at p_0 [16, 17] (see also [18] for a more recent discussion). Such a system is characterized by the conditions imposed at p_0 :

$$\begin{aligned} K_{I_1 \dots I_n \bar{J}} &= K_{I \bar{J}_1 \dots \bar{J}_n} = 0, & n > 1, \\ K_{I_1 \dots I_n} &= K_{\bar{J}_1 \dots \bar{J}_n} = 0, \\ K_{I\bar{J}} &= \delta_{I\bar{J}}. \end{aligned} \quad (2.15)$$

The specific feature of Hermitian symmetric spaces is that, in addition, more conditions hold:

$$K_{I_1 \dots I_m \bar{J}_1 \dots \bar{J}_n} = 0, \quad m \neq n. \quad (2.16)$$

These conditions imply that the Kähler potential is invariant under arbitrary $U(1)$ phase transformations, $K(e^{i\alpha} \Phi, e^{-i\alpha} \bar{\Phi}) = K(\Phi, \bar{\Phi})$, and therefore $K(\Phi, \bar{\Phi}) = F(\Phi \bar{\Phi})$, with $F(\Phi \bar{\Phi})$ a real analytic function.

In accordance with [10, 11], for any Hermitian symmetric space \mathcal{M} one can find the solution $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ to the equations of motion (2.8) in a closed form. Let $\Upsilon_0(\zeta)$ denote the value of $\Upsilon_*(\zeta)$ at the origin of the Kähler normal coordinate system, $\Upsilon_0(\zeta) = \Upsilon_*(\zeta; \Phi = 0, \bar{\Phi} = 0, \Sigma_0, \bar{\Sigma}_0)$, with Σ_0 a tangent vector at the origin. It is easy to check that

$$\Upsilon_0(\zeta) = \Sigma_0 \zeta, \quad \check{\Upsilon}_0(\zeta) = -\frac{\bar{\Sigma}_0}{\zeta} \quad (2.17)$$

solve the equations (2.8) at $\Phi = 0$ and respect the initial conditions. A next step is to distribute this solution to any point Φ of the manifold \mathcal{M} , that is to make use of $\Upsilon_*(\zeta; \Phi = 0, \bar{\Phi} = 0, \Sigma_0, \bar{\Sigma}_0)$ in order to obtain $\Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$.

Let G be the isometry group of the Hermitian symmetric space \mathcal{M} . It acts transitively on \mathcal{M} by holomorphic transformations. Without loss of generality, we can always choose the open domain U , on which the Kähler normal coordinate system is defined, to be simply connected. Then, we can construct a coset representative, $\mathcal{S}: U \rightarrow G$, defined to be a holomorphic isometry transformation $\mathcal{S}(p): \mathcal{M} \rightarrow \mathcal{M}$ such that

$$\mathcal{S}(p)p_0 = p, \quad \mathcal{S}(p) \in G,$$

for any point $p \in U$. In other words, $\mathcal{S}(p)$ maps the origin to p . In local coordinates, $\mathcal{S}(p) = \mathcal{S}(\Phi, \bar{\Phi})$, and it acts on a generic point $q \in U$ parametrized by complex variables $(\Psi^I, \bar{\Psi}^{\bar{J}})$ as follows:

$$\Psi \rightarrow \Psi' = f(\Psi; \Phi, \bar{\Phi}), \quad f(0; \Phi, \bar{\Phi}) = \Phi. \quad (2.18)$$

Now, we should point out that the holomorphic isometry transformations leave the equations (2.8) invariant. This means that applying the group transformation $\mathcal{S}(\Phi, \bar{\Phi})$ to $\Upsilon_0(\zeta)$, eq. (2.17), gives

$$\Upsilon_0(\zeta) \rightarrow \Upsilon_*(\zeta) = f(\Upsilon_0(\zeta); \Phi, \bar{\Phi}) = f(\Sigma_0 \zeta; \Phi, \bar{\Phi}), \quad \Upsilon_*(0) = \Phi. \quad (2.19)$$

Imposing the second initial condition in (2.9),

$$\Sigma^I = \Sigma_0^J \frac{\partial}{\partial \Psi^J} f^I(\Psi; \Phi, \bar{\Phi}) \Big|_{\Psi=0}, \quad (2.20)$$

we are in a position to uniquely express Σ_0 in terms of Σ and $\Phi, \bar{\Phi}$. By construction, Σ is a complex linear superfield constrained as in (1.2). As to Σ_0 , it obeys a generalized linear constraint that follows from (2.20) by requiring $\bar{D}^2 \Sigma = 0$.

Our consideration shows that $\Upsilon_*(\zeta)$ is independent of $\bar{\Sigma}$, i.e. $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma)$, for all Hermitian symmetric spaces. The same conclusion also follows from the fact that, in the case of Hermitian symmetric spaces, the algebraic equations of motion are equivalent to the holomorphic geodesic equation (with complex evolution parameter) [10, 11]

$$\frac{d^2 \Upsilon_*^I(\zeta)}{d\zeta^2} + \Gamma_{JK}^I(\Upsilon_*(\zeta), \bar{\Phi}) \frac{d\Upsilon_*^J(\zeta)}{d\zeta} \frac{d\Upsilon_*^K(\zeta)}{d\zeta} = 0, \tag{2.21}$$

under the same initial conditions (2.9). Here $\Gamma_{JK}^I(\Phi, \bar{\Phi})$ are the Christoffel symbols for the Kähler metric $g_{I\bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$.

A crucial element in the above scheme, for the Hermitian symmetric spaces, is the coset representative $\mathcal{S}(\Phi, \bar{\Phi})$. There is huge freedom in its choice, since it can always be replaced by $\mathcal{S}(\Phi, \bar{\Phi}) \rightarrow \mathcal{S}(\Phi, \bar{\Phi}) h(\Phi, \bar{\Phi})$, with $h(\Phi, \bar{\Phi})$ an arbitrary function taking its values in the stability group H of the origin, $\Phi = 0$. It is extremely important to use this freedom to choose the “correct” coset representative, since our final aim is to compute the tangent bundle action

$$S_{\text{tb}}[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}] = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^8 z K(\Upsilon_*(\zeta), \check{\Upsilon}_*(\zeta)) . \tag{2.22}$$

With a complicated coset representative chosen, it will be practically impossible to do the contour integral on the right. In what follows, we will construct such a “correct” coset representatives for four series of compact Hermitian symmetric spaces, and then extend the results to the non-compact case.

3. Algebraic setup: compact case

This section is devoted to the construction of coset representatives for the four series of irreducible compact Hermitian symmetric spaces.

3.1 The symmetric space $U(n+m)/U(n) \times U(m)$

The complex Grassmannian $G_{m,n+m}(\mathbb{C}) = U(n+m)/U(n) \times U(m)$ is defined to be the space of m -planes through the origin in \mathbb{C}^{n+m} . Its elements can be considered to be the equivalence classes of complex $(n+m) \times m$ matrices of rank m ,

$$x = (x^I{}_{\beta}) = \begin{pmatrix} x^i{}_{\beta} \\ x_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \hat{x} \end{pmatrix}, \quad i = 1, \dots, n \quad \alpha, \beta = 1, \dots, m \tag{3.1}$$

defined modulo arbitrary transformations of the form

$$x \rightarrow x g, \quad g \in \text{GL}(m, \mathbb{C}) . \tag{3.2}$$

By applying such a transformation one can turn x into a matrix u constrained by

$$u^\dagger u = \mathbb{1}_m . \tag{3.3}$$

In other words, the m vectors $u_\beta = (u^I_\beta)$ form an orthonormal basis on the m -plane. Then, the ‘gauge’ freedom (3.2) reduces to

$$u \rightarrow u g, \quad g \in U(m). \quad (3.4)$$

Consider the open domain in $G_{m,n+m}(\mathbb{C})$ singled out by the condition $\det \hat{u} \neq 0$ (an open coordinate chart in the Grassmann space). Then, we can uniquely represent

$$\hat{u} = s h, \quad s^\dagger = s = (s_{\alpha\beta}), \quad h = (h^\alpha_\beta) \in U(m), \quad (3.5)$$

with s being a positive definite Hermitian matrix. Equivalently we have

$$\hat{u} \hat{u}^\dagger = s^2, \quad \hat{u}^\dagger \hat{u} = h^{-1} s^2 h. \quad (3.6)$$

Eq. (3.3) becomes

$$\tilde{u}^\dagger \tilde{u} + \hat{u}^\dagger \hat{u} = \tilde{u}^\dagger \tilde{u} + h^{-1} s^2 h = \mathbb{1}_m. \quad (3.7)$$

It is worth pointing out that the ‘gauge’ freedom (3.4) can be completely fixed by setting $h = \mathbb{1}_m$.

Introduce an Hermitian $(n+m) \times (n+m)$ matrix $F(u)$,

$$F = \begin{pmatrix} x \mathbb{1}_n + \tilde{u} \Lambda \tilde{u}^\dagger & \tilde{u} h^{-1} \\ h \tilde{u}^\dagger & s \end{pmatrix} = F^\dagger, \quad \Lambda = \lambda(h^{-1} s h) = h^{-1} \lambda(s) h, \quad \Lambda^\dagger = \Lambda. \quad (3.8)$$

Here x is a real number, and $\lambda(s)$ some function. We require $F(u)$ to be unitary,

$$F^\dagger F = \mathbb{1}_{n+m}.$$

This can be shown to hold if

$$\lambda(s) = -\frac{x \mathbb{1}_m + s}{\mathbb{1}_m - s^2}, \quad x^2 = 1. \quad (3.9)$$

We have to choose

$$x = -1 \quad \longrightarrow \quad \lambda(s) = \frac{\mathbb{1}_m}{\mathbb{1}_m + s}, \quad (3.10)$$

in order for $\lambda(s)$ to be well defined at $s_0 = \mathbb{1}_m$. Here s_0 corresponds to

$$u_0 = \begin{pmatrix} \tilde{u}_0 \\ \hat{u}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{1}_m \end{pmatrix}. \quad (3.11)$$

The crucial property of $F(u)$ is that it maps u_0 to the equivalence class containing u :

$$F(u) u_0 = \begin{pmatrix} \tilde{u} h^{-1} \\ s \end{pmatrix} \sim \begin{pmatrix} \tilde{u} \\ \hat{u} \end{pmatrix} = u. \quad (3.12)$$

Let us point out that the matrix $F(u)$ is invariant under the ‘gauge’ transformations (3.4).

It is useful to replace $F(u)$ by

$$G(u) = F(u) \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_m \end{pmatrix} = \begin{pmatrix} \mathbb{1}_n - \tilde{u}h^{-1}\lambda(s)h\tilde{u}^\dagger & \tilde{u}h^{-1} \\ -h\tilde{u}^\dagger & s \end{pmatrix}. \quad (3.13)$$

This matrix has the property that $G(u_0) = \mathbb{1}_{n+m}$. In the case $m = 1$, the matrix G is unimodular, $G(u) \in \text{SU}(n+1)$, as can be checked using the identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det D. \quad (3.14)$$

Similar arguments can be used to show $G(u) \in \text{SU}(n+m)$ in the case $n \geq m$.

Let us introduce local complex coordinates, $\Phi = (\Phi^{i\alpha})$, in the Grassmann manifold

$$u = \begin{pmatrix} \tilde{u} \\ \hat{u} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{u} \hat{u}^{-1} \\ \mathbb{1}_m \end{pmatrix} = \begin{pmatrix} \tilde{u} h^{-1} s^{-1} \\ \mathbb{1}_m \end{pmatrix} \equiv \begin{pmatrix} \Phi \\ \mathbb{1}_m \end{pmatrix}. \quad (3.15)$$

Eq. (3.7) is equivalent to

$$\Phi^\dagger \Phi + \mathbb{1}_m = s^{-2}. \quad (3.16)$$

By construction, the variables Φ are invariant under the transformations (3.4). Since the coset representative (3.13) is also invariant under (3.4), the matrix elements of $G(u)$ depend solely on Φ and its conjugate:⁴

$$G(u) = G(\Phi, \bar{\Phi}) = \begin{pmatrix} \mathbb{1}_n - \Phi s \lambda(s) s \Phi^\dagger & \Phi s \\ -s \Phi^\dagger & s \end{pmatrix}. \quad (3.17)$$

Given an element of the isometry group,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{U}(n+m), \quad (3.18)$$

it acts on a generic point (in the coordinate chart) of the Grassmann space

$$v = \begin{pmatrix} \tilde{v} \\ \hat{v} \end{pmatrix} \sim \begin{pmatrix} z \\ \mathbb{1}_m \end{pmatrix} \quad (3.19)$$

by the holomorphic fractional linear transformation

$$\begin{pmatrix} z \\ \mathbb{1}_m \end{pmatrix} \rightarrow \begin{pmatrix} z' \\ \mathbb{1}_m \end{pmatrix}, \quad z' = (Az + B)(Cz + D)^{-1}. \quad (3.20)$$

Choosing here $g = G(\Phi, \bar{\Phi})$ gives the action of the coset representative on the manifold.

⁴In the case $m = 1$, coset representative (3.17) reduces to that used in [11] to derive the tangent bundle formulation for the $\mathcal{N} = 2$ supersymmetric sigma model (2.1) associated with $\mathbb{C}P^n$.

Keeping in mind subsequent applications, let us describe a slightly different form for the coset representative (3.17). Along with the matrix s , eq. (3.16), we can introduce

$$\underline{s}^2 = \frac{\mathbb{1}_n}{\Phi\Phi^\dagger + \mathbb{1}_n}, \quad (3.21)$$

with the properties

$$\Phi s = \underline{s} \Phi, \quad \Phi^\dagger \underline{s} = s \Phi^\dagger. \quad (3.22)$$

Then, the coset representative (3.17) can be rewritten as follows:

$$G(\Phi, \bar{\Phi}) = \begin{pmatrix} \underline{s} & \Phi s \\ -\Phi^\dagger \underline{s} & s \end{pmatrix}. \quad (3.23)$$

This coset representative is well-known in the literature, see e.g. [29], and can be viewed as a generalization of the Wigner construction [30] used in his classification of the unitary representations of the Poincaré group.

3.2 The symmetric spaces $SO(2n)/U(n)$ and $Sp(n)/U(n)$

As is known [31, 32] (see also [33] for a related discussion), the Hermitian symmetric spaces $SO(2n)/U(n)$ and $Sp(n)/U(n)$ can be realized as quadrics in the Grassmannian $G_{n,2n}(\mathbb{C})$,

$$x^T \mathbf{J}_\epsilon x = 0, \quad (3.24)$$

where

$$\mathbf{J}_\epsilon = \begin{pmatrix} 0 & \mathbb{1}_n \\ \epsilon \mathbb{1}_n & 0 \end{pmatrix}, \quad \epsilon = \begin{cases} +, & \text{for } SO(2n)/U(n), \\ -, & \text{for } Sp(n)/U(n). \end{cases} \quad (3.25)$$

The submanifold (3.24) is invariant under the action of $O(2n)$ for $\epsilon = +1$, and $Sp(n)$ for $\epsilon = -1$, with these groups realized as follows:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2n), \quad g^T \mathbf{J}_\epsilon g = \mathbf{J}_\epsilon. \quad (3.26)$$

The two conditions (3.26) imply

$$\bar{A} = D, \quad \bar{C} = \epsilon B, \quad (3.27)$$

with \bar{F} denoting the complex conjugate of a matrix F .

In the complex local coordinates (3.15), eq. (3.24) takes the form

$$\Phi^T + \epsilon \Phi = 0. \quad (3.28)$$

In this case, eq. (3.16) can be rewritten as

$$\mathbb{1}_n + \Phi^\dagger \Phi = \mathbb{1}_n - \epsilon \bar{\Phi} \Phi = s^{-2}, \quad (3.29)$$

and for the matrix \underline{s} , eq. (3.21), we obtain

$$\underline{s} = \bar{s} . \tag{3.30}$$

Finally, the coset representative (3.23) turns into

$$G_\epsilon(\Phi, \bar{\Phi}) = \begin{pmatrix} \bar{s} & \Phi s \\ \epsilon \bar{\Phi} \bar{s} & s \end{pmatrix}, \quad G_\epsilon(\Phi, \bar{\Phi}) \in \begin{cases} \text{U}(2n) \cap \text{SO}(2n), & \epsilon = 1, \\ \text{U}(2n) \cap \text{Sp}(n), & \epsilon = -1. \end{cases} \tag{3.31}$$

Its crucial property is that it maps

$$\begin{pmatrix} 0 \\ \mathbb{1}_n \end{pmatrix} \rightarrow G(\Phi, \bar{\Phi}) \begin{pmatrix} 0 \\ \mathbb{1}_n \end{pmatrix} = \begin{pmatrix} \Phi s \\ s \end{pmatrix} \sim \begin{pmatrix} \Phi \\ \mathbb{1}_n \end{pmatrix} . \tag{3.32}$$

3.3 The symmetric space $\text{SO}(n+2)/\text{SO}(n) \times \text{SO}(2)$

In accordance with [32, 34], the Hermitian symmetric space⁵ $\text{SO}(n+2)/\text{SO}(n) \times \text{SO}(2)$ is holomorphically equivalent to the complex quadric hypersurface $Q_n(\mathbb{C})$ in the projective space $\mathbb{C}P^{n+1}$, and real-analytically isomorphic to the oriented Grassmann manifold $\tilde{G}_{2,n+2}(\mathbb{R})$. Let us recall the relevant geometric constructions.

Consider the projective space $\mathbb{C}P^{n+1} = G_{1,n+2}(\mathbb{C})$. Its elements are non-zero complex $(n+2)$ -vectors,

$$Z = (Z^I) \neq 0, \quad I = 1, \dots, n+2, \tag{3.33}$$

defined modulo the equivalence relation

$$Z \sim \lambda Z, \quad \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\} . \tag{3.34}$$

The complex quadric $Q_n(\mathbb{C})$ is the following hypersurface in $\mathbb{C}P^{n+1}$:

$$Q_n(\mathbb{C}) = \left\{ Z^T Z = (Z^1)^2 + (Z^2)^2 + \dots + (Z^{n+2})^2 = 0, \quad Z \in \mathbb{C}P^{n+1} \right\} . \tag{3.35}$$

Let X and Y be the real and imaginary parts of Z , respectively,

$$Z = X + iY . \tag{3.36}$$

On the quadric surface, $Z \in Q_n(\mathbb{C})$, the real $(n+2)$ -vectors X and Y obey the relations

$$X^T X = Y^T Y \neq 0, \quad X^T Y = Y^T X = 0 . \tag{3.37}$$

In other words, considered as element of Euclidean space \mathbb{R}^{n+2} , the non-zero vectors X and Y have the same length and are orthogonal to each other. Therefore, they are linearly independent. At this point, it is useful to introduce the $(n+2) \times 2$ matrix of rank 2:

$$x = (x^I_\beta), \quad X = (x^I_1), \quad Y = (x^I_2) . \tag{3.38}$$

⁵This space is irreducible for $n > 2$.

Clearly, this matrix defines a two-plane through the origin in \mathbb{R}^{n+2} . Now, the relations (3.37) can be rewritten as

$$x^T x = \alpha \mathbb{1}_2, \quad \alpha = \frac{1}{2} \text{tr}(x^T x) > 0, \quad (3.39)$$

while the equivalence relation (3.34) turns into

$$x \sim x g, \quad g \in \mathbb{R}^+ \times \text{SO}(2), \quad (3.40)$$

with \mathbb{R}^+ the multiplicative group of positive real numbers.

The real realization for $Q_n(\mathbb{C})$ considered above, makes obvious a relationship of this manifold to the oriented Grassmannian $\tilde{G}_{2,n+2}(\mathbb{R})$ – the space of oriented two-planes through the origin in \mathbb{R}^{n+2} . The elements of $\tilde{G}_{2,n+2}(\mathbb{R})$ can be identified with the equivalence classes of real $(n+2) \times 2$ matrices of rank 2,

$$x = (x^I{}_\beta) = \begin{pmatrix} x^i{}_\beta \\ x_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \hat{x} \end{pmatrix}, \quad i = 1, \dots, n \quad \alpha, \beta = 1, 2 \quad (3.41)$$

with respect to the equivalence relation

$$x \sim x g, \quad g \in \text{GL}^+(2, \mathbb{R}). \quad (3.42)$$

Indeed, since x is of rank 2, the matrix $x^T x$ is positive definite. Then, by applying a transformation of the form $x \rightarrow x g$, with $g \in \text{GL}^+(2, \mathbb{R})$, one can always make x obey eq. (3.39). Hence, we can identify $Q_n(\mathbb{C})$ with $\tilde{G}_{2,n+2}(\mathbb{R})$.

Given a matrix $x \in \tilde{G}_{2,n+2}(\mathbb{R})$, its equivalence class contains a matrix

$$u = (u^I{}_\beta) = \begin{pmatrix} \tilde{u} \\ \hat{u} \end{pmatrix} \quad (3.43)$$

constrained by

$$u^T u = \tilde{u}^T \tilde{u} + \hat{u}^T \hat{u} = \mathbb{1}_2. \quad (3.44)$$

Such a matrix defines an orthonormal basis on the two-plane chosen. In what follows, we deal with such matrices only, within each equivalence class, when studying various aspects of the Grassmann manifold. Under eq. (3.44), the equivalence relation (3.42) reduces to

$$u \sim u g, \quad g \in \text{SO}(2). \quad (3.45)$$

Let $U_{n+1,n+2}$ be the open domain in $\tilde{G}_{2,n+2}(\mathbb{R})$ singled out by the condition $\det \hat{u} \neq 0$ (an open coordinate chart in the Grassmann manifold). It consists of the two components with empty intersection: (i) $U_{n+1,n+2}^{(+)}$ in which $\det \hat{u} > 0$; and (ii) $U_{n+1,n+2}^{(-)}$ in which $\det \hat{u} < 0$. They are mapped on each other, say, by a rotation through angle π in the $(n, n+1)$ plane in \mathbb{R}^{n+2} . For our purposes, it will be sufficient to consider the chart $U_{n+1,n+2}^{(+)}$ only. Then, we can represent

$$\hat{u} = s h, \quad s = s^T = (s_{\alpha\beta}), \quad h \in \text{SO}(2), \quad (3.46)$$

with s positive definite.

Given a matrix $u \in U_{n+1,n+2}^{(+)}$, we can associate with it the following $SO(n+2)$ -transformation:

$$G(u) = \begin{pmatrix} \mathbb{1}_n - \tilde{u}h^{-1}\lambda(s)h\tilde{u}^T & \tilde{u}h^{-1} \\ -h\tilde{u}^T & s \end{pmatrix}, \quad \lambda(s) = \frac{\mathbb{1}_2}{\mathbb{1}_2 + s}. \quad (3.47)$$

The crucial property of $G(u)$ is

$$G(u)u_0 = \begin{pmatrix} \tilde{u}h^{-1} \\ s \end{pmatrix}, \quad u_0 = \begin{pmatrix} \tilde{u}_0 \\ \hat{u}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{1}_2 \end{pmatrix}. \quad (3.48)$$

It is time to let the complex structure, which is intrinsically defined on $Q_n(\mathbb{C})$, enter the scene. Let us proceed by introducing a complex $(n+2)$ -vector of the form

$$w = (w^I) := (u^I_1 + i u^I_2) = \begin{pmatrix} \tilde{w} \\ \hat{w} \end{pmatrix} \equiv \begin{pmatrix} \varphi \\ z \end{pmatrix}, \quad \varphi = (\varphi^i), \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (3.49)$$

Then, the relations encoded in (3.44) are rewritten as

$$w^T w = \varphi^T \varphi + (z_1)^2 + (z_2)^2 = 0, \quad (3.50)$$

$$w^\dagger w = \varphi^\dagger \varphi + |z_1|^2 + |z_2|^2 = 2. \quad (3.51)$$

The equivalence relation (3.45) turns into

$$w \sim e^{i\sigma} w, \quad \sigma \in \mathbb{R}. \quad (3.52)$$

In what follows, we choose the gauge condition $h = \mathbb{1}_2$. Recall that $s = s^T$ is positive definite, that is

$$s = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}, \quad s_{11} > 0, \quad s_{22} > 0, \quad s_{11}s_{22} - (s_{12})^2 > 0. \quad (3.53)$$

These results tell us that both the components of z ,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} (sh)_{11} + i(sh)_{12} \\ (sh)_{12} + i(sh)_{22} \end{pmatrix} = \begin{pmatrix} s_{11} + i s_{12} \\ s_{12} + i s_{22} \end{pmatrix}, \quad (3.54)$$

are non-vanishing, $z_{1,2} \neq 0$. If we further introduce new variables

$$z_\pm = z_1 \pm i z_2, \quad (3.55)$$

then one obtains⁶

$$z_- = \overline{z_+} = s_{11} + s_{22} > 0, \quad \left| \frac{z_+}{z_-} \right|^2 < 1. \quad (3.56)$$

⁶In deriving eq. (3.56), we have fixed the gauge freedom (3.52) by imposing the condition $h = \mathbb{1}_2$. In general, the expression for z_- is as follows: $z_- = e^{i\sigma}(s_{11} + s_{22})$, with σ a real parameter.

Finally, introducing projective variables

$$\Phi = \frac{\varphi}{z_-}, \quad \rho = \frac{z_+}{z_-}, \quad (3.57)$$

the equations (3.50) and (3.51) turn into

$$\Phi^T \Phi + \rho = 0, \quad (3.58)$$

$$2\Phi^\dagger \Phi + 1 + |\rho|^2 = \frac{4}{(z_-)^2}. \quad (3.59)$$

Eq. (3.56) and (3.58) tell us

$$|\Phi^T \Phi| < 1. \quad (3.60)$$

Let us consider an isometry transformation $g \in \text{SO}(n+2)$.

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad g^T g = \mathbb{1}. \quad (3.61)$$

The linear action $u \rightarrow u' = gu$ induces the holomorphic fractional linear transformation

$$\begin{aligned} \Phi &\rightarrow \Phi' = \left\{ (1, -i) \left(C \Phi + D \Gamma(\Phi) \right) \right\}^{-1} \left\{ A \Phi + B \Gamma(\Phi) \right\}, \\ \Gamma(\Phi) &= \frac{1}{2} \begin{pmatrix} 1 - \Phi^T \Phi \\ i(1 + \Phi^T \Phi) \end{pmatrix}. \end{aligned} \quad (3.62)$$

Here we have used the fact that z_- transforms as follows:

$$\frac{z'_-}{z_-} = (1, -i) \left(C \Phi + D \Gamma(\Phi) \right) \equiv e^{\Lambda(\Phi)}. \quad (3.63)$$

Unlike z_- , its transform z'_- is no longer real, but its phase is a gauge degree of freedom.

The Kähler potential [35–37] is

$$K(\Phi, \bar{\Phi}) = \frac{1}{2} \ln \left(1 + 2\Phi^\dagger \Phi + |\Phi^T \Phi|^2 \right). \quad (3.64)$$

Under the holomorphic isometry transformation (3.62), it changes as

$$K(\Phi', \bar{\Phi}') = K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}), \quad (3.65)$$

with $\Lambda(\Phi)$ given in (3.63). This can be seen from the identity

$$1 + 2\Phi'^\dagger \Phi' + |\rho'|^2 = \left(1 + 2\Phi^\dagger \Phi + |\rho|^2 \right) \left| \frac{z_-}{z'_-} \right|^2, \quad (3.66)$$

in conjunction with eq. (3.63).

Let us turn to the problem of expressing the coset representative (3.47) in terms of the complex coordinates introduced above. In the gauge $h = \mathbb{1}_2$ we can rewrite $G(u)$ as

$$G(\Phi, \bar{\Phi}) = \begin{pmatrix} \underline{s} & \tilde{u} \\ -\tilde{u}^T & s \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad (3.67)$$

where

$$\underline{s}^2 = \mathbb{1}_n - \tilde{u} \tilde{u}^T, \quad s^2 = \mathbb{1}_2 - \tilde{u}^T \tilde{u}. \quad (3.68)$$

For the matrix blocks in (3.67) we then get

$$\begin{aligned} \mathcal{A} &= \sqrt{\mathbb{1}_n - \frac{z_-^2}{2}(\Phi\Phi^\dagger + \bar{\Phi}\bar{\Phi}^T)}, \\ \mathcal{B} &= \frac{1}{2}z_-(\Phi, \bar{\Phi})\gamma, \quad \mathcal{C} = -\frac{1}{2}z_-\gamma^\dagger \begin{pmatrix} \Phi^\dagger \\ \Phi^T \end{pmatrix}, \\ \mathcal{D} &= \frac{1}{2}\gamma^\dagger \sqrt{\mathbb{1}_2 - \frac{z_-^2}{2}\Delta}\gamma, \end{aligned} \quad (3.69)$$

where

$$\gamma = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Phi^\dagger\Phi & \bar{\Phi}^T\bar{\Phi} \\ \Phi^T\Phi & \Phi^\dagger\Phi \end{pmatrix}. \quad (3.70)$$

One can easily check that \mathcal{D} can be rewritten as

$$\mathcal{D} = \frac{1}{4}z_-\gamma^\dagger F\gamma, \quad F = \begin{pmatrix} 1 & -\bar{\Phi}^T\bar{\Phi} \\ -\Phi^T\Phi & 1 \end{pmatrix}. \quad (3.71)$$

Furthermore it is useful to verify the following relations

$$\begin{pmatrix} \Phi^\dagger \\ \Phi^T \end{pmatrix} \mathcal{A} = \frac{1}{2}\gamma\mathcal{D}\gamma^\dagger \begin{pmatrix} \Phi^\dagger \\ \Phi^T \end{pmatrix} = \frac{1}{2}z_-F \begin{pmatrix} \Phi^\dagger \\ \Phi^T \end{pmatrix}, \quad (3.72)$$

and

$$\mathcal{A}^T\mathcal{A} = \mathbb{1}_n - z_-^2(\Phi, \bar{\Phi}) \begin{pmatrix} \Phi^\dagger \\ \Phi^T \end{pmatrix}. \quad (3.73)$$

Equation (3.59) gives the expression for z_- in terms of Φ and its conjugate. The isometry transformation $G(\Phi, \bar{\Phi}) \in \text{SO}(n+2)$ maps the origin, $\Phi_0 = 0$, to the point Φ . On a generic point Υ of the symmetric space, it acts by the rule:

$$\Upsilon \rightarrow \Upsilon' = \left\{ (1, -i) \left(\mathcal{C}\Upsilon + \mathcal{D}\Gamma(\Upsilon) \right) \right\}^{-1} \left\{ \mathcal{A}\Upsilon + \mathcal{B}\Gamma(\Upsilon) \right\}, \quad (3.74)$$

with the two-vector $\Gamma(\Upsilon)$ defined similarly to (3.62).

4. The (co)tangent bundle over $U(n+m)/U(n) \times U(m)$

Here we apply the procedure described in section 2 to the case of Grassmann manifolds $G_{m,n+m}(\mathbb{C}) = U(n+m)/U(n) \times U(m)$. In accordance with section 2, the tangent bundle action is

$$S = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^8z K(\Upsilon, \tilde{\Upsilon}), \quad (4.1)$$

with $K(\Phi, \bar{\Phi})$ the Kähler potential.

In the case of the Grassmannian $G_{m,n+m}(\mathbb{C})$, the Kähler potential (see, e.g. [38]) is

$$K(\Phi, \Phi^\dagger) = \ln \det(\mathbb{1}_m + \Phi^\dagger \Phi) = \ln \det(\mathbb{1}_n + \Phi \Phi^\dagger), \quad (4.2)$$

where $\Phi = (\Phi^{i\alpha})$ and $\Phi^\dagger = (\bar{\Phi}^{\bar{\alpha}i})$, with $\bar{\Phi}^{\bar{\alpha}i} = \overline{\Phi^{i\alpha}}$. It will be assumed that the indices can be raised and lowered using the ‘flat metrics’ $\delta_{\alpha\bar{\beta}}$ and $\delta_{i\bar{j}}$, and their inverses, in particular $\bar{\Phi}^{\bar{\alpha}i} = \bar{\Phi}_\alpha^i = \bar{\Phi}_{\alpha i} = \bar{\Phi}^{\bar{\alpha}}_i$. The Kähler metric can readily be shown to be

$$\begin{aligned} g_{i\alpha, \bar{j}\bar{\beta}} &= \left(\frac{\mathbb{1}_m}{\mathbb{1}_m + \Phi^\dagger \Phi} \right)_{\alpha\bar{\beta}} \left\{ \delta_{\bar{j}i} - \left(\Phi \frac{\mathbb{1}_m}{\mathbb{1}_m + \Phi^\dagger \Phi} \Phi^\dagger \right)_{\bar{j}i} \right\} \\ &= \left(\frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi \Phi^\dagger} \right)_{\bar{j}i} \left\{ \delta_{\alpha\bar{\beta}} - \left(\Phi^\dagger \frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi \Phi^\dagger} \Phi \right)_{\alpha\bar{\beta}} \right\} \\ &= \left(\frac{\mathbb{1}_m}{\mathbb{1}_m + \Phi^\dagger \Phi} \right)_{\alpha\bar{\beta}} \left(\frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi \Phi^\dagger} \right)_{\bar{j}i}, \end{aligned} \quad (4.3)$$

where we have used the identities

$$\frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi \Phi^\dagger} \Phi = \Phi \frac{\mathbb{1}_m}{\mathbb{1}_m + \Phi^\dagger \Phi}, \quad \Phi^\dagger \frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi \Phi^\dagger} = \frac{\mathbb{1}_m}{\mathbb{1}_m + \Phi^\dagger \Phi} \Phi^\dagger. \quad (4.4)$$

With the choice

$$K(\Upsilon, \check{\Upsilon}) = \ln \det(\mathbb{1}_m + \check{\Upsilon}^T \Upsilon) = \ln \det(\mathbb{1}_n + \Upsilon \check{\Upsilon}^T) \quad (4.5)$$

in action (4.1), the equations of motion for the auxiliary superfields are

$$\oint \frac{d\zeta}{\zeta} \zeta^n \left(\mathbb{1}_m + \check{\Upsilon}_*^T \Upsilon_* \right)^{-1} \check{\Upsilon}_*^T = 0, \quad n \geq 2. \quad (4.6)$$

As explained in section 2, one can easily check that (2.17) solve the equations (4.6). According to (2.19), we can obtain the solution at any point of the base manifold. Acting by the coset representative (3.13) on (2.17), we obtain

$$\Upsilon_* = \left\{ (\mathbb{1}_n - \Phi \lambda s^2 \Phi^\dagger) \Sigma_0 \zeta + \Phi s \right\} (-s \Phi^\dagger \Sigma_0 \zeta + s)^{-1}. \quad (4.7)$$

From here we read off the tangent vector at Φ

$$\begin{aligned} \Sigma &\equiv \left. \frac{\partial \Upsilon_*}{\partial \zeta} \right|_{\zeta=0} = (\mathbb{1}_n - \Phi \lambda s^2 \Phi^\dagger + \Phi s \Phi^\dagger) \Sigma_0 s^{-1} = (\mathbb{1}_n + \Phi \lambda s \Phi^\dagger) \Sigma_0 s^{-1} \\ &= \underline{\underline{\Sigma}}^{-1} \Sigma_0 s^{-1}. \end{aligned} \quad (4.8)$$

This result allows us to express Σ_0 in terms of Σ . Let us substitute the solution (4.7) into the potential (4.5). Then we have

$$\begin{aligned} K(\Upsilon_*, \check{\Upsilon}_*) &= \ln \det \left(\mathbb{1}_m + \check{\Upsilon}_*^T \Upsilon_* \right) \\ &= \ln \det \left(\mathbb{1}_m + \Phi^\dagger \Phi - s^{-1} \Sigma_0^\dagger \Sigma_0 s^{-1} \right) \\ &\quad - \ln \det \left(\mathbb{1}_m + \frac{1}{\zeta} s^{-1} \Sigma_0^\dagger \Phi s \right) - \ln \det \left(\mathbb{1}_m - s \Phi^\dagger \Sigma_0 s^{-1} \zeta \right). \end{aligned} \quad (4.9)$$

Here we have used eqs. (3.10) and (3.16), and their corollary

$$\Phi s^2 \Phi^\dagger + (\mathbb{1}_n - \Phi \lambda s^2 \Phi^\dagger)^2 = \mathbb{1}_n. \quad (4.10)$$

The expression in the last line of (4.9) does not contribute to the action (4.1) where the ζ integral only singles out the constant part, and it will not be written down explicitly in what follows. Now, eq. (4.8) implies $s^{-1} \Sigma_0^\dagger = \Sigma^\dagger \underline{s}$ and $\Sigma_0 s^{-1} = \underline{s} \Sigma$, and hence

$$\begin{aligned} K(\Upsilon_*, \check{\Upsilon}_*) &= \ln \det \left(\mathbb{1}_m + \Phi^\dagger \Phi - \Sigma^\dagger (\mathbb{1}_n + \Phi \Phi^\dagger)^{-1} \Sigma \right) + \dots \\ &= K(\Phi, \Phi^\dagger) + \ln \det \left(\mathbb{1}_m - (\mathbb{1}_m + \Phi^\dagger \Phi)^{-1} \Sigma^\dagger (\mathbb{1}_n + \Phi \Phi^\dagger)^{-1} \Sigma \right) + \dots, \end{aligned} \quad (4.11)$$

with $K(\Phi, \Phi^\dagger)$ the Kähler potential of the base manifold, eq. (4.2). Evaluated at Υ_* and $\check{\Upsilon}_*$, the action (4.1) turns into the tangent bundle action

$$\begin{aligned} S &= \int d^8 z \left\{ K(\Phi, \Phi^\dagger) + \ln \det \left(\mathbb{1}_m - (\mathbb{1}_m + \Phi^\dagger \Phi)^{-1} \Sigma^\dagger (\mathbb{1}_n + \Phi \Phi^\dagger)^{-1} \Sigma \right) \right\} \\ &= \int d^8 z \left\{ K(\Phi, \Phi^\dagger) + \ln \det \left(\mathbb{1}_n - (\mathbb{1}_n + \Phi \Phi^\dagger)^{-1} \Sigma (\mathbb{1}_m + \Phi^\dagger \Phi)^{-1} \Sigma^\dagger \right) \right\}. \end{aligned} \quad (4.12)$$

It is not difficult to see that

$$\begin{aligned} I &= \text{tr} \ln \left(\mathbb{1}_m - (\mathbb{1}_m + \Phi^\dagger \Phi)^{-1} \Sigma^\dagger (\mathbb{1}_n + \Phi \Phi^\dagger)^{-1} \Sigma \right) \\ &= \text{tr} \ln \left(\mathbb{1}_n - (\mathbb{1}_n + \Phi \Phi^\dagger)^{-1} \Sigma (\mathbb{1}_m + \Phi^\dagger \Phi)^{-1} \Sigma^\dagger \right) \end{aligned} \quad (4.13)$$

is actually a scalar field on an open domain of the zero section of the tangent bundle. By construction, Σ defines a holomorphic tangent vector with world indices. Instead of using the coordinate basis, we can decompose tangent vectors with respect to the vielbein defined in (A.2),

$$\Sigma \rightarrow \tilde{\Sigma} = \underline{s} \Sigma s, \quad \Sigma^\dagger \rightarrow \tilde{\Sigma}^\dagger = s \Sigma^\dagger \underline{s}. \quad (4.14)$$

Then we readily obtain

$$I = \text{tr} \ln \left(\mathbb{1}_m - \tilde{\Sigma}^\dagger \tilde{\Sigma} \right) = \text{tr} \ln \left(\mathbb{1}_n - \tilde{\Sigma} \tilde{\Sigma}^\dagger \right). \quad (4.15)$$

Our consideration shows that the tangent bundle action

$$\begin{aligned} S &= \int d^8 z \left\{ K(\Phi, \Phi^\dagger) + \text{tr} \ln \left(\mathbb{1}_m - \tilde{\Sigma}^\dagger \tilde{\Sigma} \right) \right\} \\ &= \int d^8 z \left\{ K(\Phi, \Phi^\dagger) + \text{tr} \ln \left(\mathbb{1}_n - \tilde{\Sigma} \tilde{\Sigma}^\dagger \right) \right\}. \end{aligned} \quad (4.16)$$

is well-defined under the following covariant conditions⁷

$$\tilde{\Sigma}^\dagger \tilde{\Sigma} < \mathbb{1}_m \iff \tilde{\Sigma} \tilde{\Sigma}^\dagger < \mathbb{1}_n. \quad (4.17)$$

⁷For an Hermitian matrix H , $H^\dagger = H$, the notation $H > 0$ means that H is positive definite.

In appendix A, the curvature tensor of the Grassmannian is computed, eq. (A.5). It follows from (A.5) that the Taylor expansion of (4.16) in powers of Σ and $\bar{\Sigma}$ can be represented in the universal form (2.11).

By comparing the equations (4.8) and (4.14), one can see that $\tilde{\Sigma}$ coincides with Σ_0 . Nevertheless, here and below we prefer to use the “tilde” notation for (co)tangent vectors decomposed vectors with respect to the vielbein (A.2).

Let us derive the cotangent bundle over the Grassmann manifold. In order to obtain it, we need to dualize the complex linear superfields $\Sigma = (\Sigma^{i\alpha})$ in (4.12) into chiral superfields $\psi = (\psi_{\alpha i})$ forming the components of a cotangent vector. To apply the relevant Legendre transformation, the action (4.12) is to be replaced by the following one

$$S = \int d^8z \left\{ K(\Phi, \Phi^\dagger) + \text{tr} \ln \left(\mathbb{1}_m - sU^\dagger \underline{s}^2 U s \right) + \frac{1}{2} \text{tr} (U\psi) + \frac{1}{2} \text{tr} (\psi^\dagger U^\dagger) \right\}, \quad (4.18)$$

where $U = (U^{i\alpha})$ is a complex unconstrained superfield. By construction, U is a tangent vector at the point Φ of the base manifold. Therefore ψ is a one-form at the same point. Varying the action with respect to ψ gives $U = \Sigma$, and then one obtains the original action (4.12). On the other hand, varying U allows one to express U in terms of Φ , ψ and their conjugates, thus ending up with a dual formulation.

In order to simplify further expressions and to make symmetry properties more transparent, it is useful to decompose the tangent and cotangent vectors with respect to the vielbein (A.2),

$$U \rightarrow \tilde{U} = \underline{s} U s, \quad \psi \rightarrow \tilde{\psi} = s^{-1} \psi \underline{s}^{-1}. \quad (4.19)$$

Then, the action becomes

$$S = \int d^8z \left\{ K(\Phi, \Phi^\dagger) + \text{tr} \ln \left(\mathbb{1}_m - \tilde{U}^\dagger \tilde{U} \right) + \frac{1}{2} \text{tr} (\tilde{U} \tilde{\psi}) + \frac{1}{2} \text{tr} (\tilde{\psi}^\dagger \tilde{U}^\dagger) \right\}. \quad (4.20)$$

We should point out that it is the variables ψ which are chiral, while their covariant counterparts, $\tilde{\psi}$, obey a generalized chirality constraint. To eliminate the auxiliary fields \tilde{U} and \tilde{U}^\dagger , we consider their equations of motion:

$$\frac{1}{2} \tilde{\psi} = (\mathbb{1}_m - \tilde{U}^\dagger \tilde{U})^{-1} \tilde{U}^\dagger = \tilde{U}^\dagger (\mathbb{1}_n - \tilde{U} \tilde{U}^\dagger)^{-1}, \quad (4.21)$$

and the conjugate equation. These lead to

$$(\mathbb{1}_n - \tilde{U} \tilde{U}^\dagger)^{-1} = \frac{1}{2} \left(\mathbb{1}_n \pm \sqrt{\mathbb{1}_n + \tilde{\psi}^\dagger \tilde{\psi}} \right).$$

We have to choose the “plus” solution in order to satisfy the requirement that $\tilde{\psi}^\dagger \tilde{\psi} \rightarrow 0$ implies $\tilde{U} \tilde{U}^\dagger \rightarrow 0$ and vice versa. Thus

$$(\mathbb{1}_n - \tilde{U} \tilde{U}^\dagger)^{-1} = \frac{1}{2} \left(\mathbb{1}_n + \sqrt{\mathbb{1}_n + \tilde{\psi}^\dagger \tilde{\psi}} \right). \quad (4.22)$$

We also readily obtain

$$\frac{1}{2} \tilde{U} \tilde{\psi} = -\frac{1}{2} \left(\mathbb{1}_n - \sqrt{\mathbb{1}_n + \tilde{\psi}^\dagger \tilde{\psi}} \right). \quad (4.23)$$

As a result the action (4.20) turns into

$$\begin{aligned}
 S &= \int d^8z \left\{ K(\Phi, \Phi^\dagger) - \text{tr} \ln \left(\mathbb{1}_n + \sqrt{\mathbb{1}_n + \tilde{\psi}^\dagger \tilde{\psi}} \right) + \text{tr} \sqrt{\mathbb{1}_n + \tilde{\psi}^\dagger \tilde{\psi}} \right\} \\
 &= \int d^8z \left\{ K(\Phi, \Phi^\dagger) - \text{tr} \ln \left(\mathbb{1}_m + \sqrt{\mathbb{1}_m + \tilde{\psi} \tilde{\psi}^\dagger} \right) + \text{tr} \sqrt{\mathbb{1}_m + \tilde{\psi} \tilde{\psi}^\dagger} \right\} . \quad (4.24)
 \end{aligned}$$

This action defines the cotangent bundle formulation for the $\mathcal{N} = 2$ supersymmetric sigma model (4.1) associated with the Grassmannian $G_{m,n+m}(\mathbb{C})$.

5. The (co)tangent bundle over $\text{SO}(2n)/\text{U}(n)$ and $\text{Sp}(n)/\text{U}(n)$

For the Hermitian symmetric spaces $\text{SO}(2n)/\text{U}(n)$ and $\text{Sp}(n)/\text{U}(n)$, the Kähler potentials are known to be

$$K(\Phi, \Phi^\dagger) = \ln \det(\mathbb{1}_n + \Phi^\dagger \Phi) = \ln \det(\mathbb{1}_n + \Phi \Phi^\dagger), \quad (5.1)$$

with the constraint (3.28) imposed on the variables $\Phi = (\Phi^{ij})$ and $\Phi^\dagger = (\bar{\Phi}^{\bar{j}\bar{i}})$, where $i, j = 1, \dots, n$. The Kähler metric can be read off as

$$g_{ik, \bar{l}\bar{j}} = \left(\frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi^\dagger \Phi} \right)_{k\bar{l}} \left(\frac{\mathbb{1}_n}{\mathbb{1}_n + \Phi \Phi^\dagger} \right)_{\bar{j}i}, \quad (5.2)$$

where we have used (4.4).

The $\mathcal{N} = 2$ supersymmetric sigma model (4.1) associated to (5.1) is generated by the Lagrangian (4.5), but now Υ has to obey the constraint

$$\Upsilon^T = -\epsilon \Upsilon . \quad (5.3)$$

The latter follows from (3.28) and, in particular, it requires

$$\Sigma^T = -\epsilon \Sigma . \quad (5.4)$$

Since the manifolds under consideration are imbedded into Grassmannians, the equations of motion for the auxiliary superfields have the form (4.6), and their solution is given by (4.8). It follows from (4.8)

$$\Sigma \equiv \left. \frac{\partial \Upsilon_*}{\partial \zeta} \right|_{\zeta=0} = \bar{s}^{-1} \Sigma_0 s^{-1}, \quad (5.5)$$

where we have used the identity (3.30) which holds for the manifolds $\text{SO}(2n)/\text{U}(n)$ and $\text{Sp}(n)/\text{U}(n)$. Requiring $\Sigma_0^T = -\epsilon \Sigma_0$, it then follows from (5.5) that Σ indeed obeys the algebraic constraint (5.4).

Now, it is obvious that the tangent bundle action for the symmetric spaces $\text{SO}(2n)/\text{U}(n)$ and $\text{Sp}(n)/\text{U}(n)$ is given by eq. (4.12) with $m = n$.

To derive the cotangent bundle formulation, we can again use the first-order action (4.18) with $m = n$ in which, however, the tangent $U = (U^{ij})$ and cotangent $\psi = (\psi_{ij})$ variables must obey the algebraic conditions

$$U^T = -\epsilon U, \quad \psi^T = -\epsilon \psi . \quad (5.6)$$

The equations of motion for U and U^\dagger should respect these symmetry conditions. At first sight, one could then think that the equation (4.21) should be modified in order to accommodate these conditions. Fortunately, the right-hand side of (4.21) automatically enjoys the desirable symmetry conditions,

$$\tilde{\psi} \equiv 2(\mathbb{1}_n - \tilde{U}^\dagger \tilde{U})^{-1} \tilde{U}^\dagger = 2\tilde{U}^\dagger(\mathbb{1}_n - \tilde{U}\tilde{U}^\dagger)^{-1} = -\epsilon \tilde{\psi}^T \tag{5.7}$$

provided U is chosen to obey the corresponding algebraic constraint in (5.6), $U^T = -\epsilon U$, and therefore $\tilde{U}^T = -\epsilon \tilde{U}$. As a result, all the steps implemented below eq. (4.21) to derive (4.24), remains valid in the case under consideration.

We conclude that the action (4.24) with $m = n$ defines the cotangent bundle formulation for the $\mathcal{N} = 2$ supersymmetric sigma model (4.1) associated with the symmetric spaces $\text{SO}(2n)/\text{U}(n)$ and $\text{Sp}(n)/\text{U}(n)$.

6. The (co)tangent bundle over $\text{SO}(n + 2)/\text{SO}(n) \times \text{SO}(2)$

For the $\mathcal{N} = 2$ supersymmetric sigma model (2.1) associated with the Hermitian symmetric space $\text{SO}(n + 2)/\text{SO}(n) \times \text{SO}(2)$, the (co)tangent bundle formulations have been studied in ref. [12]. The approach of [12] was based on implementing the following steps: (i) construct a coset representative in the case $n = 2$; (ii) apply it to construct the corresponding solution $\Upsilon_*(\zeta)$; (iii) make use of the latter in order to guess the explicit form of $\Upsilon_*(\zeta)$ for $n > 2$.

In this section, we are going to address the same problem by different means – the coset representative (3.67) allows us to carry out the scheme described in section 2 for general n . As is explained below, this leads to somewhat different conclusions for the cotangent bundle formulation.

The Kähler potential of the quadric surface [35 – 37] is⁸

$$K(\Phi, \bar{\Phi}) = \frac{1}{2} \ln \frac{4}{z_-^2}, \quad z_-^2 = \frac{4}{1 + 2\Phi^\dagger \Phi + |\Phi^T \Phi|^2}. \tag{6.1}$$

Its $\mathcal{N} = 2$ extension is given by

$$K(\Upsilon, \check{\Upsilon}) = \frac{1}{2} \ln \left(1 + 2\check{\Upsilon}^T \Upsilon + \Upsilon^T \Upsilon \check{\Upsilon}^T \check{\Upsilon} \right). \tag{6.2}$$

From here we read off the equations of motion for the auxiliary superfields

$$0 = \oint \frac{d\zeta}{\zeta} \zeta^n \frac{\check{\Upsilon} + \Upsilon \check{\Upsilon}^T \check{\Upsilon}}{1 + 2\check{\Upsilon}^T \Upsilon + \Upsilon^T \Upsilon \check{\Upsilon}^T \check{\Upsilon}}, \quad n \geq 2. \tag{6.3}$$

The solution to (6.3) is obtained from (3.74) by replacing $\Upsilon \rightarrow \Sigma_0 \zeta$ and $\Upsilon' \rightarrow \Upsilon_*(\zeta)$, with Σ_0 a tangent vector at $\Phi = 0$. Then, we have

$$\Upsilon_*(\zeta) = \frac{\Phi + (2/z_-) \mathcal{A} \Sigma_0 \zeta - \bar{\Phi} \Sigma_0^T \Sigma_0 \zeta^2}{1 - 2\Phi^\dagger \Sigma_0 \zeta + \overline{\Phi^T \Phi} \Sigma_0^T \Sigma_0 \zeta^2}. \tag{6.4}$$

⁸Taking the normalization $r \rightarrow r/\sqrt{2}$ in ref. [12] gives our normalization.

For the tangent vector Σ at Φ , we then obtain

$$\Sigma \equiv \left. \frac{\partial \Upsilon_*}{\partial \zeta} \right|_{\zeta=0} = \frac{2}{z_-} \mathcal{A} \Sigma_0 + 2\Phi(\Phi^\dagger \Sigma_0). \quad (6.5)$$

Using eq. (6.5), one can express Σ_0 via Σ with the aid of relations (3.71) and (3.72). One derives

$$\begin{aligned} \Phi^\dagger \Sigma_0 &= \frac{1}{4} z_-^2 \left\{ \Phi^\dagger \Sigma + \overline{\Phi^T \Phi} \Phi^T \Sigma \right\}, \\ \Phi^T \Sigma_0 &= \frac{1}{4} z_-^2 \left\{ (1 + 2|\Phi|^2) \Phi^T \Sigma - \Phi^T \Phi \Phi^\dagger \Sigma \right\}, \\ \Sigma_0^T \Sigma_0 &= \frac{1}{4} z_-^2 \Sigma^T \Sigma, \\ \Sigma_0^\dagger \Sigma_0 &= \frac{1}{4} z_-^2 \Sigma^\dagger \Sigma + \frac{1}{8} z_-^4 \left\{ (1 + 2|\Phi|^2) \Phi^T \Sigma \Sigma^\dagger \bar{\Phi} \right. \\ &\quad \left. - \overline{\Phi^T \Phi} \Phi^T \Sigma \Sigma^\dagger \Phi - \Phi^T \Phi \Phi^\dagger \Sigma \Sigma^\dagger \bar{\Phi} - \Phi^\dagger \Sigma \Sigma^\dagger \Phi \right\}. \end{aligned} \quad (6.6)$$

These relations lead to the final form for the solution Υ_* :

$$\Upsilon_*(\zeta) = \frac{\Phi + \zeta \Sigma - (z_-^2/2) \left\{ \zeta \Phi(\Phi^\dagger \Sigma + \overline{\Phi^T \Phi} \Phi^T \Sigma) + \frac{1}{2} \zeta^2 \bar{\Phi} \Sigma^T \Sigma \right\}}{1 - \frac{1}{2} z_-^2 \left\{ \zeta(\Phi^\dagger \Sigma + \overline{\Phi^T \Phi} \Phi^T \Sigma) - \frac{1}{2} \zeta^2 \overline{\Phi^T \Phi} \Sigma^T \Sigma \right\}}. \quad (6.7)$$

Now let us turn to calculating the Lagrangian

$$L = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} K(\Upsilon_*, \check{\Upsilon}_*). \quad (6.8)$$

Considerations similar to those used to derive eq. (3.66) give

$$\begin{aligned} K(\Upsilon_*, \check{\Upsilon}_*) &= K(\Phi, \bar{\Phi}) + \frac{1}{2} \ln \left(1 - 2\Sigma_0^\dagger \Sigma_0 + |\Sigma_0^T \Sigma_0|^2 \right) - \frac{1}{2} \ln x(\zeta) - \frac{1}{2} \ln \check{x}(\zeta), \\ x(\zeta) &= 1 - 2\Phi^\dagger \Sigma_0 \zeta + \overline{\Phi^T \Phi} \Sigma_0^T \Sigma_0 \zeta^2. \end{aligned} \quad (6.9)$$

The last two terms in the expression for $K(\Upsilon_*, \check{\Upsilon}_*)$ do not contribute to the Lagrangian upon the integration over ζ . Using the third and fourth formulas in (6.6), we obtain

$$\begin{aligned} L &= K(\Phi, \bar{\Phi}) + \frac{1}{2} \ln \left(1 - \frac{1}{2} z_-^2 \Sigma^\dagger \Sigma - \frac{1}{4} z_-^4 \left\{ (1 + 2|\Phi|^2) \Phi^T \Sigma \Sigma^\dagger \bar{\Phi} \right. \right. \\ &\quad \left. \left. - \overline{\Phi^T \Phi} \Phi^T \Sigma \Sigma^\dagger \Phi - \Phi^T \Phi \Phi^\dagger \Sigma \Sigma^\dagger \bar{\Phi} - \Phi^\dagger \Sigma \Sigma^\dagger \Phi + \frac{1}{16} z_-^4 |\Sigma^T \Sigma|^2 \right\} \right). \end{aligned} \quad (6.10)$$

The Lagrangian can be rewritten in a geometric form. In order to do that, we use the metric of the compact quadric surface

$$g_{i\bar{j}} = \frac{1}{4} z_-^2 \delta_{i\bar{j}} + \frac{1}{8} z_-^4 \left\{ \Phi^i \bar{\Phi}^{\bar{j}} (1 + 2|\Phi|^2) - \bar{\Phi}^{\bar{i}} \Phi^j - \Phi^i \Phi^{\bar{j}} (\overline{\Phi^T \Phi}) - \bar{\Phi}^{\bar{i}} \bar{\Phi}^{\bar{j}} (\Phi^T \Phi) \right\}. \quad (6.11)$$

From this we have

$$\begin{aligned} g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} &= \frac{1}{4} z_-^2 |\Sigma|^2 + \frac{1}{8} z_-^4 \left\{ (1 + 2|\Phi|^2) |\Phi^T \Sigma|^2 - |\Phi^\dagger \Sigma|^2 \right. \\ &\quad \left. - (\Phi^T \Phi)(\Phi^\dagger \bar{\Sigma})(\Phi^\dagger \Sigma) - (\overline{\Phi^T \Phi})(\Phi^T \Sigma)(\Phi^T \bar{\Sigma}) \right\}. \end{aligned} \quad (6.12)$$

Thus

$$L = K(\Phi, \bar{\Phi}) + \frac{1}{2} \ln \left(1 - 2g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} + \frac{1}{16} z_-^4 |\Sigma^T \Sigma|^2 \right). \quad (6.13)$$

Here the second term is a scalar field on the tangent bundle. Therefore, the combination $z_-^4 |\Sigma^T \Sigma|^2$ must be a scalar constructed in terms of the tangent vector Σ^i and its conjugate, the metric $g_{i\bar{j}}$ and the Riemann curvature $R_{i\bar{j}k\bar{l}} \equiv \partial_k \bar{\partial}_{\bar{l}} g_{i\bar{j}} - g^{m\bar{n}} \partial_m g_{i\bar{j}} \bar{\partial}_{\bar{n}} g_{k\bar{l}}$, with $\partial_i = \partial / \partial \Phi^i$. It is sufficient to determine such an expression at any given point of the base manifold, say at $\Phi = 0$, since the base manifold is a symmetric space. This gives

$$2(g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}})^2 + \frac{1}{2} R_{i\bar{j}k\bar{l}} \Sigma^i \bar{\Sigma}^{\bar{j}} \Sigma^k \bar{\Sigma}^{\bar{l}} = \frac{1}{16} z_-^4 |\Sigma^T \Sigma|^2. \quad (6.14)$$

As a result, we arrive at the tangent bundle action [12]

$$S = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) + \frac{1}{2} \ln \left(1 - 2g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} + 2(g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}})^2 + \frac{1}{2} R_{i\bar{j}k\bar{l}} \Sigma^i \bar{\Sigma}^{\bar{j}} \Sigma^k \bar{\Sigma}^{\bar{l}} \right) \right\}. \quad (6.15)$$

Note that the tangent vector in (6.15) should be constrained as follows:

$$\begin{aligned} 1 - 2g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} + 2(g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}})^2 + \frac{1}{2} R_{i\bar{j}k\bar{l}} \Sigma^i \bar{\Sigma}^{\bar{j}} \Sigma^k \bar{\Sigma}^{\bar{l}} &> 0, \\ g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} &< 1. \end{aligned} \quad (6.16)$$

To make the action (6.15) well-defined, we actually need only the first constraint in (6.16). The latter can be shown to imply $g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} \neq 1$, and therefore its space of solutions consists of two connected components. The second constraint in (6.16) picks up one of the components.

Finally, it remains to dualize the tangent bundle action (6.15) in order to generate the cotangent bundle formulation. The derivation is very similar to that performed for the non-compact quadric surface in section 10 and appendix B. Here we only give the result:

$$\begin{aligned} S = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) - \frac{1}{2} \ln \left(\Lambda + \sqrt{2(\Lambda + g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)} \right) + \frac{1}{4} \left(\Lambda + \sqrt{2(\Lambda + g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)} \right) \right. \\ \left. + \frac{1}{2} \frac{(g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)^2 + \frac{1}{4} R^{\bar{i}j\bar{k}l} \bar{\psi}_{\bar{i}} \psi_j \bar{\psi}_{\bar{k}} \psi_l}{\Lambda + \sqrt{2(\Lambda + g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)}} \right\}, \end{aligned} \quad (6.17)$$

where

$$\Lambda = 1 + \sqrt{1 + 2g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j + 2(g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)^2 + \frac{1}{2} R^{\bar{i}j\bar{k}l} \bar{\psi}_{\bar{i}} \psi_j \bar{\psi}_{\bar{k}} \psi_l}. \quad (6.18)$$

The dualization was also studied in ref. [12]. The derivation was performed in the case $n = 2$ case and it was claimed that the cotangent bundle action should be valid for general n case since it is written in geometric terms. In the present paper, the cotangent bundle action is derived for general n , and its explicit form is different from that obtained in ref. [12].

7. Algebraic setup: non-compact case

This section is devoted to the construction of coset representatives for the four series of non-compact Hermitian symmetric spaces.

7.1 The symmetric space $U(n, m)/U(n) \times U(m)$

The non-compact Hermitian symmetric space $U(n, m)/U(n) \times U(m)$ can be identified with an open subset of $G_{m, n+m}(\mathbb{C})$. This subset consists of those m -planes in \mathbb{C}^{n+m} which obey the equation

$$x^\dagger \Omega x > 0, \quad \Omega = \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_m \end{pmatrix}. \quad (7.1)$$

Here x is a complex $(n+m) \times m$ matrix of rank m ,

$$x = (x^I{}_\beta) = \begin{pmatrix} x^i{}_\beta \\ x_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \hat{x} \end{pmatrix}, \quad i = 1, \dots, n \quad \alpha, \beta = 1, \dots, m \quad (7.2)$$

which is defined modulo arbitrary transformations of the form

$$x \rightarrow x g, \quad g \in \text{GL}(m, \mathbb{C}). \quad (7.3)$$

One of the consequences of eq. (7.1) is that $\hat{x} \in \text{GL}(m, \mathbb{C})$.

By applying a transformation of the form (7.3), one can turn x into a matrix u under the equation

$$u^\dagger \Omega u = \hat{u}^\dagger \hat{u} - \tilde{u}^\dagger \tilde{u} = \mathbb{1}_m \quad \longrightarrow \quad \det \hat{u} \neq 0. \quad (7.4)$$

With such a choice, the ‘gauge’ freedom (7.3) reduces to

$$u \rightarrow u g, \quad g \in U(m). \quad (7.5)$$

We further represent \hat{u} according to eq. (3.5), with s being a uniquely chosen positive definite Hermitian matrix. Then, eq. (7.4) becomes

$$\hat{u}^\dagger \hat{u} - \tilde{u}^\dagger \tilde{u} = h^{-1} s^2 h - \tilde{u}^\dagger \tilde{u} = \mathbb{1}_m. \quad (7.6)$$

Let us introduce

$$G(u) = \begin{pmatrix} \mathbb{1}_n + \tilde{u} h^{-1} \lambda(s) h \tilde{u}^\dagger & \tilde{u} h^{-1} \\ h \tilde{u}^\dagger & s \end{pmatrix}, \quad \lambda(s) = \frac{\mathbb{1}_m}{\mathbb{1}_m + s}. \quad (7.7)$$

The matrix $G(u)$ has the properties

$$G^\dagger(u) = G(u), \quad G(u) \in \text{SU}(n, m). \quad (7.8)$$

Another crucial feature of $G(u)$ is that it enjoys the property (3.12), with u_0 defined in (3.11). In other words, $G(u)$ is a global coset representative for $U(n, m)/U(n) \times U(m)$.

Global complex coordinates on $U(n, m)/U(n) \times U(m)$

$$u = \begin{pmatrix} \tilde{u} \\ \hat{u} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{u} \hat{u}^{-1} \\ \mathbb{1}_m \end{pmatrix} = \begin{pmatrix} \tilde{u} h^{-1} s^{-1} \\ \mathbb{1}_m \end{pmatrix} \equiv \begin{pmatrix} \Phi \\ \mathbb{1}_m \end{pmatrix}. \quad (7.9)$$

Eq. (7.6) is equivalent to

$$\mathbb{1}_m - \Phi^\dagger \Phi = s^{-2}. \quad (7.10)$$

Since $s > 0$, we equivalently have

$$\Phi^\dagger \Phi < \mathbb{1}_m. \quad (7.11)$$

This relation defines a classical bounded symmetric domain [40, 31]. Now, the coset representative (7.7) turns into

$$G(\Phi, \bar{\Phi}) = \begin{pmatrix} \mathbb{1}_n + \Phi s \lambda(s) s \Phi^\dagger & \Phi s \\ s \Phi^\dagger & s \end{pmatrix}. \quad (7.12)$$

The coset representative obtained can also be rewritten in the form

$$G(\Phi, \bar{\Phi}) = \begin{pmatrix} \underline{s} & \Phi s \\ \Phi^\dagger \underline{s} & s \end{pmatrix}, \quad (7.13)$$

where the matrices s and \underline{s} are defined as

$$s^2 = \frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi^\dagger \Phi}, \quad \underline{s}^2 = \frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi \Phi^\dagger}, \quad \Phi s^2 = \underline{s}^2 \Phi, \quad s^2 \Phi^\dagger = \Phi^\dagger \underline{s}^2. \quad (7.14)$$

7.2 The symmetric spaces $SO^*(2n)/U(n)$ and $Sp(n, \mathbb{R})/U(n)$

Following [31], the Hermitian symmetric spaces $SO^*(2n)/U(n)$ and $Sp(n, \mathbb{R})/U(n)$ can be identified with special open domains in $G_{n, 2n}(\mathbb{C})$ consisting of those n -planes in \mathbb{C}^{2n} which obey the constraints

$$x^\dagger \Omega x > 0, \quad x^T \mathbf{J}_\epsilon x = 0, \quad \Omega = \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}. \quad (7.15)$$

Here the matrix \mathbf{J}_ϵ is defined in (3.25), with $\epsilon = +1$ corresponding to $SO^*(2n)$, and $\epsilon = -1$ to $Sp(n, \mathbb{R})$. It is pertinent to recall the definition of the groups $SO^*(2n)$ and $Sp(n, \mathbb{R})$ (see, e.g. [39]):

$$G = \left\{ g \in SU(n, n), \quad g^T \mathbf{J}_\epsilon g = \mathbf{J}_\epsilon \right\}, \quad \epsilon = \begin{cases} +1, & \text{for } SO^*(2n), \\ -1, & \text{for } Sp(n, \mathbb{R}). \end{cases} \quad (7.16)$$

We can introduce global complex coordinates on the manifold,

$$X \sim \begin{pmatrix} \Phi \\ \mathbb{1}_n \end{pmatrix}, \quad \Phi^T + \epsilon \Phi = 0, \quad (7.17)$$

with the $n \times n$ matrix Φ constrained as

$$\Phi^\dagger \Phi < \mathbb{1}_n . \quad (7.18)$$

In the case $n = m$, if we choose Φ in (7.13) to be antisymmetric or symmetric, $\Phi^T = -\epsilon\Phi$, then $\underline{s} = \bar{s}$, with \bar{s} the complex conjugate of s . As a result, the coset representative (7.13) becomes

$$G_\epsilon(\Phi, \bar{\Phi}) = \begin{pmatrix} \bar{s} & \Phi s \\ -\epsilon \bar{\Phi} \bar{s} & s \end{pmatrix}, \quad G_\epsilon(\Phi, \bar{\Phi}) \in \begin{cases} \text{SO}^*(2n), & \epsilon = +1, \\ \text{Sp}(n, \mathbb{R}), & \epsilon = -1. \end{cases} \quad (7.19)$$

7.3 The symmetric space $\text{SO}_0(n, 2)/\text{SO}(n) \times \text{SO}(2)$

As a real manifold, the Hermitian symmetric space⁹ $\text{SO}_0(n, 2)/\text{SO}(n) \times \text{SO}(2)$ can be identified [35] with an open subset of the oriented Grassmann manifold $\tilde{G}_{2, n+2}(\mathbb{R})$. In the notation of subsection 3.3, see eqs. (3.41) and (3.42), consider the following domain in $\tilde{G}_{2, n+2}(\mathbb{R})$:

$$\mathcal{M} = \left\{ x \in \tilde{G}_{2, n+2}(\mathbb{R}), \quad x^T \Omega x > 0 \right\}, \quad \Omega = \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}. \quad (7.20)$$

By construction, this domain is a transformation space of the group $\text{O}(n, 2)$. It follows from (7.20) that the 2×2 block \hat{x} is non-singular, $\hat{x} \in \text{GL}(2, \mathbb{R})$. As a topological space, the domain \mathcal{M} consists of two connected components with empty intersection, $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, defined as

$$\mathcal{M}^+ = \left\{ x \in \mathcal{M}, \quad \det \hat{x} > 0 \right\}; \quad \mathcal{M}^- = \left\{ x \in \mathcal{M}, \quad \det \hat{x} < 0 \right\}. \quad (7.21)$$

With respect to the action of the subgroup $\text{SO}_0(n, 2) \in \text{O}(n, 2)$, the sub-domains \mathcal{M}^+ and \mathcal{M}^- can be shown to be the orbits. We identify the Hermitian symmetric space $\text{SO}_0(n, 2)/\text{SO}(n) \times \text{SO}(2)$ with the orbit \mathcal{M}^+ .

For $x \in \mathcal{M}^+$, its equivalence class contains a matrix

$$u = (u^I_\alpha) = \begin{pmatrix} \tilde{u} \\ \hat{u} \end{pmatrix} \quad (7.22)$$

constrained to obey

$$u^T \Omega u = \hat{u}^T \hat{u} - \tilde{u}^T \tilde{u} = \mathbb{1}_2, \quad \det \hat{u} > 0. \quad (7.23)$$

We can further represent

$$\hat{u} = s h, \quad s = s^T = (s_{\alpha\beta}), \quad h \in \text{SO}(2), \quad (7.24)$$

with s positive definite. Under eq. (7.23), the ‘gauge’ freedom (3.42), reduces to

$$u \sim u g, \quad g \in \text{SO}(2). \quad (7.25)$$

⁹Here $\text{SO}_0(n, 2)$ denotes the connected component of the identity in $\text{O}(n, 2)$.

This residual freedom can be completely fixed by choosing in (7.24) $h = \mathbb{1}_2$.

Let us introduce the coset representative

$$G(u) = \begin{pmatrix} \mathbb{1}_n + \tilde{u}h^{-1}\lambda(s)h\tilde{u}^T & \tilde{u}h^{-1} \\ h\tilde{u}^T & s \end{pmatrix}, \quad \lambda(s) = \frac{\mathbb{1}_2}{\mathbb{1}_2 + s}. \quad (7.26)$$

The crucial property of $G(u)$ is

$$G(u)u_0 = \begin{pmatrix} \tilde{u}h^{-1} \\ s \end{pmatrix}, \quad u_0 = \begin{pmatrix} \tilde{u}_0 \\ \hat{u}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{1}_2 \end{pmatrix}. \quad (7.27)$$

Introduce a complex $(n+2)$ -vector

$$w = (w^I) := (u^I_1 + i u^I_2) = \begin{pmatrix} \tilde{w} \\ \hat{w} \end{pmatrix} \equiv \begin{pmatrix} \varphi \\ z \end{pmatrix}, \quad \varphi = (\varphi^i), \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (7.28)$$

Now, the first equation in (7.23) is equivalent to

$$w^T \Omega w = -\varphi^T \varphi + (z_1)^2 + (z_2)^2 = 0, \quad (7.29)$$

$$w^\dagger \Omega w = -\varphi^\dagger \varphi + |z_1|^2 + |z_2|^2 = 2. \quad (7.30)$$

The ‘gauge’ freedom (7.25) becomes

$$w \sim e^{i\sigma} w, \quad \sigma \in \mathbb{R}. \quad (7.31)$$

In what follows, we choose the gauge condition $h = \mathbb{1}_2$. Recall that $s = s^T$ is positive definite, that is

$$s = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}, \quad s_{11} > 0, \quad s_{22} > 0, \quad s_{11}s_{22} - (s_{12})^2 > 0. \quad (7.32)$$

These results tell us that both the components of z ,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} (sh)_{11} + i(sh)_{12} \\ (sh)_{12} + i(sh)_{22} \end{pmatrix} = \begin{pmatrix} s_{11} + i s_{12} \\ s_{12} + i s_{22} \end{pmatrix}, \quad (7.33)$$

are non-vanishing, $z_{1,2} \neq 0$. If we further introduce new variables

$$z_\pm = z_1 \pm i z_2, \quad (7.34)$$

then one readily sees¹⁰

$$z_- = \bar{z}_- = s_{11} + s_{22} > 0, \quad \left| \frac{z_+}{z_-} \right|^2 < 1. \quad (7.35)$$

Finally, introducing projective variables

$$\Phi = \frac{\varphi}{z_-}, \quad \rho = \frac{z_+}{z_-}, \quad (7.36)$$

¹⁰In deriving eq. (7.35), we have fixed the gauge freedom (7.31) by imposing the condition $h = \mathbb{1}_2$. In general, the expression for z_- is as follows: $z_- = e^{i\sigma}(s_{11} + s_{22})$, with σ a real parameter.

the equations (7.29) and (7.30) turn into

$$-\Phi^T \Phi + \rho = 0, \tag{7.37}$$

$$-2\Phi^\dagger \Phi + 1 + |\rho|^2 = \frac{4}{(z_-)^2}. \tag{7.38}$$

In conjunction with eq. (7.35), we now see that the n complex variables $\Phi = (\Phi^i)$ span the domain

$$-2\Phi^\dagger \Phi + 1 + |\Phi^T \Phi|^2 > 0, \quad |\Phi^T \Phi| < 1. \tag{7.39}$$

These conditions define a classical bounded symmetric domain [40, 35, 36].

Let us consider an isometry transformation $g \in \text{SO}_0(n, 2)$.

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad g^T \Omega g = \Omega, \tag{7.40}$$

or equivalently

$$A^T A - C^T C = \mathbb{1}_n, \quad D^T D - B^T B = \mathbb{1}_2, \quad A^T B = C^T D. \tag{7.41}$$

The fact that g belongs to the connected component of the identity in $\text{O}(n, 2)$, is expressed as follows:

$$\det A > 0, \quad \det D > 0. \tag{7.42}$$

The linear action $u \rightarrow u' = gu$ induces the holomorphic fractional linear transformation

$$\begin{aligned} \Phi &\rightarrow \Phi' = \left\{ (1, -i) \left(C \Phi + D \Gamma(\Phi) \right) \right\}^{-1} \left\{ A \Phi + B \Gamma(\Phi) \right\}, \\ \Gamma(\Phi) &= \frac{1}{2} \begin{pmatrix} 1 + \Phi^T \Phi \\ i(1 - \Phi^T \Phi) \end{pmatrix}. \end{aligned} \tag{7.43}$$

Here we have used the fact that z_- transforms as follows:

$$\frac{z'_-}{z_-} = (1, -i) \left(C \Phi + D \Gamma(\Phi) \right) \equiv e^{\Lambda(\Phi)}. \tag{7.44}$$

Unlike z_- , its transform z'_- is no longer real, but its phase is a gauge degree of freedom.

The Kähler potential [35, 36, 41] is

$$K(\Phi, \bar{\Phi}) = -\frac{1}{2} \ln \left(1 - 2\Phi^\dagger \Phi + |\Phi^T \Phi|^2 \right). \tag{7.45}$$

Under the holomorphic isometry transformation (7.43), it changes as

$$K(\Phi', \bar{\Phi}') = K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}), \tag{7.46}$$

with $\Lambda(\Phi)$ given in (7.44). This can be seen from the identity

$$1 - 2\Phi'^\dagger \Phi' + |\rho'|^2 = \left(1 - 2\Phi^\dagger \Phi + |\rho|^2 \right) \left| \frac{z_-}{z'_-} \right|^2, \tag{7.47}$$

in conjunction with eq. (7.44).

Let us turn to the problem of expressing the coset representative (7.26) in terms of the complex coordinates introduced above. In the gauge $h = \mathbb{1}_2$ we can rewrite $G(u)$ as

$$G(\Phi, \bar{\Phi}) = \begin{pmatrix} s & \tilde{u} \\ \tilde{u}^\top & s \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad (7.48)$$

where

$$\underline{s}^2 = \mathbb{1}_n + \tilde{u} \tilde{u}^\top, \quad s^2 = \mathbb{1}_2 + \tilde{u}^\top \tilde{u}. \quad (7.49)$$

For the matrix blocks in (7.48) we then get

$$\begin{aligned} \mathcal{A} &= \sqrt{\mathbb{1}_n + \frac{z_-^2}{2}(\Phi\Phi^\dagger + \bar{\Phi}\Phi^\top)}, \\ \mathcal{B} &= \frac{1}{2}z_-(\Phi, \bar{\Phi})\gamma, \quad \mathcal{C} = \frac{1}{2}z_-\gamma^\dagger \begin{pmatrix} \Phi^\dagger \\ \Phi^\top \end{pmatrix}, \\ \mathcal{D} &= \frac{1}{2}\gamma^\dagger \sqrt{\mathbb{1}_2 + \frac{z_-^2}{2}\Delta}\gamma, \end{aligned} \quad (7.50)$$

where

$$\gamma = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Phi^\dagger\Phi & \overline{\Phi^\top\Phi} \\ \Phi^\top\Phi & \Phi^\dagger\Phi \end{pmatrix}. \quad (7.51)$$

Eq. (7.38) gives the expression for z_- in terms of Φ and its conjugate. The isometry transformation $G(\Phi, \bar{\Phi}) \in \text{SO}_0(n, 2)$ maps the origin, $\Phi_0 = 0$, to the point Φ . On a generic point Υ of the symmetric space, it acts by the rule:

$$\Upsilon \rightarrow \Upsilon' = \left\{ (1, -i)(\mathcal{C}\Upsilon + \mathcal{D}\Gamma(\Upsilon)) \right\}^{-1} \left\{ \mathcal{A}\Upsilon + \mathcal{B}\Gamma(\Upsilon) \right\}, \quad (7.52)$$

with the two-vector $\Gamma(\Upsilon)$ defined similarly to (7.43).

It is easy to check that the matrix \mathcal{D} in (7.50) possesses the following equivalent representation:

$$\mathcal{D} = \frac{1}{4}z_-\gamma^\dagger F \gamma, \quad F = \begin{pmatrix} 1 & \overline{\Phi^\top\Phi} \\ \Phi^\top\Phi & 1 \end{pmatrix}. \quad (7.53)$$

One can also directly verify that

$$\begin{pmatrix} \Phi^\dagger \\ \Phi^\top \end{pmatrix} \mathcal{A} = \frac{1}{2}\gamma \mathcal{D} \gamma^\dagger \begin{pmatrix} \Phi^\dagger \\ \Phi^\top \end{pmatrix} = \frac{1}{2}z_- F \begin{pmatrix} \Phi^\dagger \\ \Phi^\top \end{pmatrix}. \quad (7.54)$$

Another useful piece of information is that the first equation in (7.41) is equivalent, in the case of $G(\Phi, \bar{\Phi})$, to

$$\mathcal{A}^\top \mathcal{A} = \mathbb{1}_n + \frac{1}{2}z_-^2(\Phi, \bar{\Phi}) \begin{pmatrix} \Phi^\dagger \\ \Phi^\top \end{pmatrix}. \quad (7.55)$$

8. The (co)tangent bundle over $U(n, m)/U(n) \times U(m)$

The Kähler potential in this cases is given as

$$K(\Phi, \Phi^\dagger) = -\ln \det(\mathbb{1}_m - \Phi^\dagger \Phi) = -\ln \det(\mathbb{1}_n - \Phi \Phi^\dagger), \quad (8.1)$$

where $\Phi = (\Phi^{i\alpha})$ and $\Phi^\dagger = (\bar{\Phi}^{\bar{\alpha}i})$. The Kähler metric can be read off to be

$$g_{i\alpha, \bar{j}\bar{\beta}} = \left(\frac{\mathbb{1}_m}{\mathbb{1}_m - \Phi^\dagger \Phi} \right)_{\alpha\bar{\beta}} \left(\frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi \Phi^\dagger} \right)_{\bar{j}i}. \quad (8.2)$$

where we have used identities similar to eq. (4.4),

$$\frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi \Phi^\dagger} \Phi = \Phi \frac{\mathbb{1}_m}{\mathbb{1}_m - \Phi^\dagger \Phi}, \quad \Phi^\dagger \frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi \Phi^\dagger} = \frac{\mathbb{1}_m}{\mathbb{1}_m - \Phi^\dagger \Phi} \Phi^\dagger. \quad (8.3)$$

The explicit structure of the Kähler potential forces us to choose

$$K(\Upsilon, \check{\Upsilon}) = -\ln \det(\mathbb{1}_m - \check{\Upsilon}^T \Upsilon) = -\ln \det(\mathbb{1}_n - \Upsilon \check{\Upsilon}^T) \quad (8.4)$$

in the action (4.1). The equations of motion for the auxiliary superfields are:

$$\oint \frac{d\zeta}{\zeta} \zeta^n (\mathbb{1}_m - \check{\Upsilon}_*^T \Upsilon_*)^{-1} \check{\Upsilon}_*^T = 0, \quad n \geq 2. \quad (8.5)$$

Their solution, $\Upsilon_*(\zeta)$, is obtained by applying the coset representative (7.12) to (2.17).

$$\Upsilon_* = \left\{ (\mathbb{1}_n + \Phi \lambda s^2 \Phi^\dagger) \Sigma_0 s^{-1} \zeta + \Phi \right\} (\mathbb{1}_m + s \Phi^\dagger \Sigma_0 s^{-1} \zeta)^{-1}. \quad (8.6)$$

From here one reads off the tangent vector at Φ

$$\begin{aligned} \Sigma \equiv \left. \frac{\partial \Upsilon_*}{\partial \zeta} \right|_{\zeta=0} &= (\mathbb{1}_n + \Phi \lambda s^2 \Phi^\dagger - \Phi s \Phi^\dagger) \Sigma_0 s^{-1} = (\mathbb{1}_n - \Phi \lambda s \Phi^\dagger) \Sigma_0 s^{-1} \\ &= \underline{s}^{-1} \Sigma_0 s^{-1}. \end{aligned} \quad (8.7)$$

This result allows us to express Σ_0 in terms of Σ . Let us substitute the solution (8.6) into the potential (8.4).

$$\begin{aligned} K(\Upsilon_*, \check{\Upsilon}_*) &= -\ln \det \left(\mathbb{1}_m - \check{\Upsilon}_*^T \Upsilon_* \right) \\ &= -\ln \det \left(\mathbb{1}_m - \Phi^\dagger \Phi + s^{-1} \Sigma_0^\dagger \Sigma_0 s^{-1} \right) \\ &\quad + \ln \det \left(\mathbb{1}_m - \frac{1}{\zeta} s^{-1} \Sigma_0^\dagger \Phi s \right) + \ln \det \left(\mathbb{1}_m + s \Phi^\dagger \Sigma_0 s^{-1} \zeta \right). \end{aligned} \quad (8.8)$$

Here we have used eqs. (7.7) and (7.10), and their corollary

$$-\Phi s^2 \Phi^\dagger + (\mathbb{1}_n + \Phi \lambda s^2 \Phi^\dagger)^2 = \mathbb{1}_n. \quad (8.9)$$

From (8.8) we obtain the tangent bundle action

$$\begin{aligned} S &= \int d^8 z \left\{ K(\Phi, \Phi^\dagger) - \ln \det \left(\mathbb{1}_m + (\mathbb{1}_m - \Phi^\dagger \Phi)^{-1} \Sigma^\dagger (\mathbb{1}_n - \Phi \Phi^\dagger)^{-1} \Sigma \right) \right\} \\ &= \int d^8 z \left\{ K(\Phi, \Phi^\dagger) - \ln \det \left(\mathbb{1}_n + (\mathbb{1}_n - \Phi \Phi^\dagger)^{-1} \Sigma (\mathbb{1}_m - \Phi^\dagger \Phi)^{-1} \Sigma^\dagger \right) \right\}, \end{aligned} \quad (8.10)$$

where $K(\Phi, \Phi^\dagger)$ is the Kähler potential, eq. (8.1). Unlike the compact case, eq. (4.17), no restrictions on the tangent variables Σ occur.

By construction, the theory with action (8.10) is invariant under the isometry group $U(n, m)$ of the base manifold. However, the symmetry properties are somewhat hidden in the action constructed. To make them manifest, it is useful to decompose the tangent vectors with respect to the vielbein (A.7):

$$\Sigma \rightarrow \tilde{\Sigma} = \underline{s} \Sigma s, \quad \Sigma^\dagger \rightarrow \tilde{\Sigma}^\dagger = s \Sigma^\dagger \underline{s}, \quad (8.11)$$

similarly to the compact case. Given an isometry transformation, it proves to act on $\tilde{\Sigma}$ as an induced local transformation from the isotropy group $U(n) \times U(m)$. Such a transformation acts on $\tilde{\Sigma}$ as follows:

$$\tilde{\Sigma} \rightarrow g_L \tilde{\Sigma} g_R, \quad g_L \in U(n), \quad g_R \in U(m). \quad (8.12)$$

This implies that the tangent bundle action

$$\begin{aligned} S &= \int d^8z \left\{ K(\Phi, \Phi^\dagger) - \text{tr} \ln \left(\mathbb{1}_m + \tilde{\Sigma}^\dagger \tilde{\Sigma} \right) \right\} \\ &= \int d^8z \left\{ K(\Phi, \Phi^\dagger) - \text{tr} \ln \left(\mathbb{1}_n + \tilde{\Sigma} \tilde{\Sigma}^\dagger \right) \right\} \end{aligned} \quad (8.13)$$

is indeed $U(n, m)$ -invariant. In appendix A, the curvature tensor of the symmetric space $U(n, m)/U(n) \times U(m)$ is computed, eq. (A.10). It follows from (A.10) that the Taylor expansion of (8.13) in powers of Σ and $\tilde{\Sigma}$ (in the domain $\tilde{\Sigma}^\dagger \tilde{\Sigma} < \mathbb{1}_m$) can be represented in the universal form (2.11).

Derivation of the cotangent bundle formulation is very similar to the compact case. One introduces the first-order action

$$S = \int d^8z \left\{ K(\Phi, \Phi^\dagger) - \text{tr} \ln \left(\mathbb{1}_m + s U^\dagger \underline{s}^2 U s \right) + \frac{1}{2} \text{tr} (U \psi) + \frac{1}{2} \text{tr} (\psi^\dagger U^\dagger) \right\}, \quad (8.14)$$

where $U = (U^{i\alpha})$ is a complex unconstrained superfield, and $\psi = (\psi_{\alpha i})$ is a chiral superfield. By construction, U is a tangent vector at the point Φ of the base manifold. Thus ψ should be a one-form at the same point. In order to derive the cotangent bundle formulation, the unconstrained superfield variables, U and U^\dagger , have to be eliminated with the aid of their equations of motion. This procedure is considerably simplified if one deals with (co)tangent vectors decomposed with respect to the vielbein (A.7),

$$U \rightarrow \tilde{U} = \underline{s} U s, \quad \psi \rightarrow \tilde{\psi} = s^{-1} \psi \underline{s}^{-1}. \quad (8.15)$$

Repeating the technical steps described in the case of the Grassmannian, we end up with the cotangent bundle action

$$\begin{aligned} S &= \int d^8z \left\{ K(\Phi, \Phi^\dagger) + \text{tr} \ln \left(\mathbb{1}_n + \sqrt{\mathbb{1}_n - \tilde{\psi}^\dagger \tilde{\psi}} \right) - \text{tr} \sqrt{\mathbb{1}_n - \tilde{\psi}^\dagger \tilde{\psi}} \right\} \\ &= \int d^8z \left\{ K(\Phi, \Phi^\dagger) + \text{tr} \ln \left(\mathbb{1}_m + \sqrt{\mathbb{1}_m - \tilde{\psi} \tilde{\psi}^\dagger} \right) - \text{tr} \sqrt{\mathbb{1}_m - \tilde{\psi} \tilde{\psi}^\dagger} \right\}. \end{aligned} \quad (8.16)$$

This action is well-defined under the following constraints:

$$\Phi^\dagger \Phi < \mathbb{1}_m, \quad \tilde{\psi}^\dagger \tilde{\psi} < \mathbb{1}_n. \quad (8.17)$$

From here it follows that the hyperkähler structure is defined on a unit ball of the zero section of the cotangent bundle over $U(n, m)/U(n) \times U(m)$.

Consider the simplest case, $n = m = 1$, analysed in [13]. This choice corresponds to the complex hyperbolic line $\mathbf{H}^1 = \text{SU}(1, 1)/\text{U}(1)$. It is known that any compact Riemann surface Σ of genus $g > 1$ can be obtained from \mathbf{H}^1 by factorization with respect to some discrete subgroups of $\text{SU}(1, 1)$, see e.g. [42]. Using the hyperkähler metric constructed on the open disc bundle in $T^*\mathbf{H}^1$, we then can generate a hyperkähler structure defined on an open neighbourhood of the zero section of the cotangent bundle $T^*\Sigma$.

9. The (co)tangent bundle over $\text{SO}^*(2n)/\text{U}(n)$ and $\text{Sp}(n, \mathbb{R})/\text{U}(n)$

For the non-compact Hermitian symmetric spaces $\text{SO}^*(2n)/\text{U}(n)$ and $\text{Sp}(n, \mathbb{R})/\text{U}(n)$, the Kähler potentials are

$$K(\Phi, \Phi^\dagger) = -\ln \det(\mathbb{1}_n - \Phi^\dagger \Phi) = -\ln \det(\mathbb{1}_n - \Phi \Phi^\dagger), \quad (9.1)$$

with $\Phi = (\Phi^{ij})$ a complex $n \times n$ matrix obeying the constraint (7.17). The Kähler metric is readily obtained to be

$$g_{ik, \bar{l}\bar{j}} = \left(\frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi^\dagger \Phi} \right)_{k\bar{l}} \left(\frac{\mathbb{1}_n}{\mathbb{1}_n - \Phi \Phi^\dagger} \right)_{\bar{j}i}, \quad (9.2)$$

where we have used eq. (8.3).

In complete analogy with the compact case, the (co)tangent bundle formulations can be deduced from those already derived for the symmetric space $U(n, m)/U(n) \times U(m)$, eqs. (8.10) and (8.16). In these actions, one should simply set $m = n$ and require the tangent linear Σ and cotangent chiral ψ variables to obey the algebraic constraints

$$\Sigma^T = -\epsilon \Sigma, \quad \psi^T = -\epsilon \psi. \quad (9.3)$$

The cotangent variables $\tilde{\psi}$ in (8.16) can now be expressed via ψ in two equivalent forms:

$$\tilde{\psi} = s^{-1} \psi \underline{s}^{-1} = s^{-1} \psi \bar{s}^{-1}. \quad (9.4)$$

One can readily fill the missing detail.

10. The (co)tangent bundle over $\text{SO}_0(n, 2)/\text{SO}(n) \times \text{SO}(2)$

In accordance with the discussion in section 7.3, the Kähler potential [35, 36, 41] is

$$K(\Phi, \bar{\Phi}) = -\frac{1}{2} \ln \frac{4}{z_-^2}, \quad \frac{4}{z_-^2} = 1 - 2\Phi^\dagger \Phi + |\Phi^T \Phi|^2. \quad (10.1)$$

Its $\mathcal{N} = 2$ extension is given by

$$K(\Upsilon, \check{\Upsilon}) = -\frac{1}{2} \ln \left(1 - 2\check{\Upsilon}^T \Upsilon + \Upsilon^T \Upsilon \check{\Upsilon}^T \check{\Upsilon} \right). \quad (10.2)$$

The equations of motion for the auxiliary superfields are

$$0 = \oint \frac{d\zeta}{\zeta} \zeta^n \frac{-\check{\Upsilon} + \Upsilon \check{\Upsilon}^T \check{\Upsilon}}{1 - 2\check{\Upsilon}^T \Upsilon + \Upsilon^T \Upsilon \check{\Upsilon}^T \check{\Upsilon}}, \quad n \geq 2. \quad (10.3)$$

Their solution, $\Upsilon_*(\zeta)$, is obtained from (7.52) by replacing $\Upsilon \rightarrow \zeta \Sigma_0$ and $\Upsilon' \rightarrow \Upsilon_*(\zeta)$. This gives

$$\Upsilon_*(\zeta) = \frac{\Phi + (2/z_-) \mathcal{A} \Sigma_0 \zeta + \bar{\Phi} \Sigma_0^T \Sigma_0 \zeta^2}{1 + 2\Phi^\dagger \Sigma_0 \zeta + \bar{\Phi}^T \bar{\Phi} \Sigma_0^T \Sigma_0 \zeta^2}. \quad (10.4)$$

From here

$$\Sigma = \left. \frac{d\Upsilon_*}{d\zeta} \right|_{\zeta=0} = \frac{2}{z_-} \mathcal{A} \Sigma_0 - 2\Phi(\Phi^\dagger \Sigma_0). \quad (10.5)$$

Now, one can express Σ_0 in (10.4) via Σ by making use of eqs. (7.54) and (7.55). One thus derives

$$\begin{aligned} \Phi^\dagger \Sigma_0 &= \frac{1}{4} z_-^2 \left\{ \Phi^\dagger \Sigma - \bar{\Phi}^T \bar{\Phi} \Phi^T \Sigma \right\}, \\ \Phi^T \Sigma_0 &= \frac{1}{4} z_-^2 \left\{ (1 - 2|\Phi|^2) \Phi^T \Sigma + \Phi^T \bar{\Phi} \Phi^\dagger \Sigma \right\}, \\ \Sigma_0^T \Sigma_0 &= \frac{1}{4} z_-^2 \Sigma^T \Sigma, \\ \Sigma_0^\dagger \Sigma_0 &= \frac{1}{4} z_-^2 \Sigma^\dagger \Sigma - \frac{1}{8} z_-^4 \left\{ (1 - 2|\Phi|^2) \Phi^T \Sigma \Sigma^\dagger \bar{\Phi} \right. \\ &\quad \left. + \bar{\Phi}^T \bar{\Phi} \Phi^T \Sigma \Sigma^\dagger \Phi + \Phi^T \bar{\Phi} \Phi^\dagger \Sigma \Sigma^\dagger \bar{\Phi} - \Phi^\dagger \Sigma \Sigma^\dagger \Phi \right\}. \end{aligned} \quad (10.6)$$

The final form for Υ_* is as follows:

$$\Upsilon_*(\zeta) = \frac{\Phi + \zeta \Sigma + (z_-^2/2) \left\{ \zeta \Phi(\Phi^\dagger \Sigma - \bar{\Phi}^T \bar{\Phi} \Phi^T \Sigma) + \frac{1}{2} \zeta^2 \bar{\Phi} \Sigma^T \Sigma \right\}}{1 + \frac{1}{2} z_-^2 \left\{ \zeta(\Phi^\dagger \Sigma - \bar{\Phi}^T \bar{\Phi} \Phi^T \Sigma) + \frac{1}{2} \zeta^2 \bar{\Phi}^T \bar{\Phi} \Sigma^T \Sigma \right\}}. \quad (10.7)$$

We now have to evaluate the superfield Lagrangian

$$L = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} K(\Upsilon_*, \check{\Upsilon}_*). \quad (10.8)$$

Considerations similar to those used to derive eq. (7.47), can be used to show that

$$\begin{aligned} K(\Upsilon_*, \check{\Upsilon}_*) &= K(\Phi, \bar{\Phi}) - \frac{1}{2} \ln \left(1 + 2\Sigma_0^\dagger \Sigma_0 + |\Sigma_0^T \Sigma_0|^2 \right) + \frac{1}{2} \ln x(\zeta) + \frac{1}{2} \ln \check{x}(\zeta), \\ x(\zeta) &= 1 + 2\Phi^\dagger \Sigma_0 \zeta + \bar{\Phi}^T \bar{\Phi} \Sigma_0^T \Sigma_0 \zeta^2. \end{aligned} \quad (10.9)$$

Clearly, the last two terms in the expression for $K(\Upsilon_*, \check{\Upsilon}_*)$ do not contribute to L . With the aid of the third and fourth relations in (10.6) we obtain

$$L = K(\Phi, \bar{\Phi}) - \frac{1}{2} \ln \left(1 + \frac{1}{2} z_-^2 \Sigma^\dagger \Sigma - \frac{1}{4} z_-^4 \left\{ (1 - 2|\Phi|^2) \Phi^T \Sigma \Sigma^\dagger \bar{\Phi} + \overline{\Phi^T \Phi} \Phi^T \Sigma \Sigma^\dagger \Phi + \Phi^T \Phi \Phi^\dagger \Sigma \Sigma^\dagger \bar{\Phi} - \Phi^\dagger \Sigma \Sigma^\dagger \Phi + \frac{1}{16} z_-^4 |\Sigma^T \Sigma|^2 \right\} \right). \quad (10.10)$$

This can be transformed to a more geometric form by taking two observations into account. First, using the expression for the metric of the non-compact quadric surface

$$g_{i\bar{j}} = \frac{1}{4} z_-^2 \delta_{i\bar{j}} - \frac{1}{8} z_-^4 \left\{ \Phi^i \bar{\Phi}^{\bar{j}} (1 - 2|\Phi|^2) - \bar{\Phi}^{\bar{i}} \Phi^j + \Phi^i \Phi^{\bar{j}} (\overline{\Phi^T \Phi}) + \bar{\Phi}^{\bar{i}} \bar{\Phi}^{\bar{j}} (\Phi^T \Phi) \right\}, \quad (10.11)$$

we find

$$g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} = \frac{1}{4} z_-^2 |\Sigma|^2 - \frac{1}{8} z_-^4 \left\{ (1 - 2|\Phi|^2) |\Phi^T \Sigma|^2 - |\Phi^\dagger \Sigma|^2 + (\Phi^T \Phi) (\Phi^\dagger \bar{\Sigma}) (\Phi^\dagger \Sigma) + (\overline{\Phi^T \Phi}) (\Phi^T \Sigma) (\Phi^T \bar{\Sigma}) \right\}. \quad (10.12)$$

Thus

$$L = K(\Phi, \bar{\Phi}) - \frac{1}{2} \ln \left(1 + 2g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} + \frac{1}{16} z_-^4 |\Sigma^T \Sigma|^2 \right). \quad (10.13)$$

Second, we note that $z_-^4 |\Sigma^T \Sigma|^2$ must be a scalar field on the tangent bundle, and therefore it can be expressed solely in terms of the tensor quantities: the holomorphic tangent vector Σ^i and its conjugate, the Riemann metric $g_{i\bar{j}}$, and finally the Riemann curvature $R_{i\bar{j}k\bar{l}} \equiv \partial_k \bar{\partial}_{\bar{l}} g_{i\bar{j}} - g^{m\bar{n}} \partial_m g_{i\bar{j}} \bar{\partial}_{\bar{n}} g_{k\bar{l}}$, with $\partial_i = \partial/\partial\Phi^i$. It is sufficient to determine such an expression at any given point of the base manifold, say at $\Phi = 0$, since the base manifold is a symmetric space. This gives

$$2(g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}})^2 - \frac{1}{2} R_{i\bar{j}k\bar{l}} \Sigma^i \bar{\Sigma}^{\bar{j}} \Sigma^k \bar{\Sigma}^{\bar{l}} = \frac{1}{16} z_-^4 |\Sigma^T \Sigma|^2. \quad (10.14)$$

As a result, we obtain the tangent bundle action

$$S = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) - \frac{1}{2} \ln \left(1 + 2g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}} + 2(g_{i\bar{j}} \Sigma^i \bar{\Sigma}^{\bar{j}})^2 - \frac{1}{2} R_{i\bar{j}k\bar{l}} \Sigma^i \bar{\Sigma}^{\bar{j}} \Sigma^k \bar{\Sigma}^{\bar{l}} \right) \right\} \quad (10.15)$$

where $K(\Phi, \bar{\Phi})$ is the Kähler potential of the non-compact quadric surface, eq. (10.1). It follows from (10.13) that the action is well-defined on the tangent bundle.

Finally, let us dualize the tangent bundle action (10.15). In order to do that we replace the action (10.15) with

$$S = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) - \frac{1}{2} \ln \left(1 + 2g_{i\bar{j}} U^i \bar{U}^{\bar{j}} + 2(g_{i\bar{j}} U^i \bar{U}^{\bar{j}})^2 - \frac{1}{2} R_{i\bar{j}k\bar{l}} U^i \bar{U}^{\bar{j}} U^k \bar{U}^{\bar{l}} \right) + \frac{1}{2} U^i \psi_i + \frac{1}{2} \bar{U}^{\bar{i}} \bar{\psi}_{\bar{i}} \right\} \quad (10.16)$$

where the tangent vectors U^i are complex unconstrained superfields, and cotangent vectors ψ_i are chiral superfields, $\bar{D}_{\bar{\alpha}} \psi_i = 0$. The variables U 's and \bar{U} 's can be eliminated with the

aid of their equations of motion. This turns the superfield Lagrangian into the hyperkähler potential

$$\begin{aligned}
 H(\Phi, \bar{\Phi}, \psi, \bar{\psi}) = & K(\Phi, \bar{\Phi}) + \frac{1}{2} \ln \left(\Lambda + \sqrt{2(\Lambda - g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)} \right) - \frac{1}{4} \left(\Lambda + \sqrt{2(\Lambda - g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)} \right) \\
 & + \frac{1}{2} \frac{(g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)^2 - \frac{1}{4} R^{\bar{i}j\bar{k}l} \bar{\psi}_{\bar{i}} \psi_j \bar{\psi}_{\bar{k}} \psi_l}{\Lambda + \sqrt{2(\Lambda - g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)}}, \tag{10.17}
 \end{aligned}$$

where

$$\Lambda = 1 + \sqrt{1 - 2g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j + 2(g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)^2 - \frac{1}{2} R^{\bar{i}j\bar{k}l} \bar{\psi}_{\bar{i}} \psi_j \bar{\psi}_{\bar{k}} \psi_l}. \tag{10.18}$$

Here the one-form variables are constrained as follows:

$$\begin{aligned}
 1 - 2g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j + 2(g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)^2 - \frac{1}{2} R^{\bar{i}j\bar{k}l} \bar{\psi}_{\bar{i}} \psi_j \bar{\psi}_{\bar{k}} \psi_l &> 0, \\
 g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j &< 1. \tag{10.19}
 \end{aligned}$$

The derivation of the above results can be found in the appendix B.

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A. Curvature tensor for Grassmannians and related symmetric spaces

Using the standard formalism of nonlinear realizations, here we compute the curvature tensor for the Grassmann manifold and symmetric spaces embedded into Grassmannians. Our consideration is a streamlined version of Hua’s analysis [31].

Using the coset representative (3.23), we obtain

$$G^{-1}dG = E + W, \tag{A.1}$$

where E is the vielbein

$$E = \begin{pmatrix} 0 & \underline{s} d\Phi s \\ -s d\Phi^\dagger \underline{s} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \mathcal{E} \\ -\mathcal{E}^\dagger & 0 \end{pmatrix}, \tag{A.2}$$

and W denotes the connection

$$W = \begin{pmatrix} \underline{s}^{-1} d\underline{s} + \underline{s} \Phi d\Phi^\dagger \underline{s} & 0 \\ 0 & s^{-1} ds + s \Phi^\dagger d\Phi s \end{pmatrix}. \tag{A.3}$$

It is easy to check that the torsion vanishes,

$$T = dE + W \wedge E + E \wedge W = 0 . \quad (\text{A.4})$$

For the curvature we get

$$\begin{aligned} R &= dW + W \wedge W \\ &= \begin{pmatrix} \underline{s} d\Phi \wedge s^2 d\Phi^\dagger \underline{s} & 0 \\ 0 & s d\Phi^\dagger \wedge \underline{s}^2 d\Phi s \end{pmatrix} = \begin{pmatrix} \mathcal{E} \wedge \mathcal{E}^\dagger & 0 \\ 0 & \mathcal{E}^\dagger \wedge \mathcal{E} \end{pmatrix} . \end{aligned} \quad (\text{A.5})$$

It is quite transparent that the above results apply to the Hermitian symmetric spaces $\text{SO}(2n)/\text{U}(n)$ and $\text{Sp}(n)/\text{U}(n)$ simply by restricting Φ to obey the appropriate algebraic symmetry conditions.

Similar calculations can be performed in the non-compact case. For the symmetric space $\text{U}(n, m)/\text{U}(n) \times \text{U}(m)$, the coset representative is given by eq. (7.13). One obtains

$$G^{-1}dG = E + W , \quad (\text{A.6})$$

where the vielbein is

$$E = \begin{pmatrix} 0 & \underline{s} d\Phi s \\ s d\Phi^\dagger \underline{s} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \mathcal{E} \\ \mathcal{E}^\dagger & 0 \end{pmatrix} , \quad (\text{A.7})$$

and the connection has the form

$$W = \begin{pmatrix} \underline{s}^{-1} d\underline{s} - \underline{s} \Phi d\Phi^\dagger \underline{s} & 0 \\ 0 & s^{-1} ds - s \Phi^\dagger d\Phi s \end{pmatrix} . \quad (\text{A.8})$$

The corresponding geometry is again torsion-free,

$$T = dE + W \wedge E + E \wedge W = 0 . \quad (\text{A.9})$$

The curvature can be shown to be

$$\begin{aligned} R &= dW + W \wedge W \\ &= \begin{pmatrix} -\underline{s} d\Phi s \wedge s d\Phi^\dagger \underline{s} & 0 \\ 0 & -s d\Phi^\dagger \underline{s} \wedge \underline{s} d\Phi s \end{pmatrix} = \begin{pmatrix} -\mathcal{E} \wedge \mathcal{E}^\dagger & 0 \\ 0 & -\mathcal{E}^\dagger \wedge \mathcal{E} \end{pmatrix} . \end{aligned} \quad (\text{A.10})$$

The results obtained for $\text{U}(n, m)/\text{U}(n) \times \text{U}(m)$ remain valid for the Hermitian symmetric spaces $\text{SO}^*(2n)/\text{U}(n)$ and $\text{Sp}(n, \mathbb{R})/\text{U}(n)$ if one restricts Φ to obey the appropriate algebraic symmetry conditions.

B. Derivation of (10.17)

This appendix is devoted to the derivation of the hyperkähler potential (10.17). Since the base manifold is a symmetric space, it is sufficient to implement the dualization, for the action (10.16), at $\Phi = 0$. The first-order Lagrangian

$$\mathcal{L} = -\frac{1}{2} \ln \Omega + \frac{1}{2} U^i \Psi_i + \frac{1}{2} \bar{U}^i \bar{\Psi}_i , \quad \Omega = 1 + 2U^\dagger U + |U^\dagger U|^2 \quad (\text{B.1})$$

leads to the following equations of motion for \bar{U} 's and U 's:

$$\frac{U^i + \bar{U}^i U^T U}{\Omega} = \frac{1}{2} \bar{\Psi}_i, \quad \frac{\bar{U}^i + U^i \bar{U}^T \bar{U}}{\Omega} = \frac{1}{2} \Psi_i, \quad (\text{B.2})$$

where Ψ is a cotangent vector at $\Phi = 0$. These equations imply

$$\frac{1}{4} \Psi^T \Psi = \frac{\bar{U}^T U}{\Omega}, \quad \frac{1}{4} \bar{\Psi}^T \bar{\Psi} = \frac{U^T U}{\Omega}, \quad (\text{B.3})$$

and also

$$\frac{1}{16} \left(1 - 2\Psi^\dagger \Psi + |\Psi^T \Psi|^2 \right) = \left(\frac{U^\dagger U}{\Omega} - \frac{1}{4} \right)^2. \quad (\text{B.4})$$

The latter is consistent if

$$1 - 2\Psi^\dagger \Psi + |\Psi^T \Psi|^2 > 0. \quad (\text{B.5})$$

By construction, the correspondence between the tangent and cotangent variables should be such that $\Upsilon \rightarrow 0 \iff \Psi \rightarrow 0$. This means that we are to choose the “minus” solution of (B.4), that is

$$\frac{U^\dagger U}{\Omega} = \frac{1}{4} \left(1 - \sqrt{1 - 2\Psi^\dagger \Psi + |\Psi^T \Psi|^2} \right). \quad (\text{B.6})$$

Now, the results obtained above can be used to express Ω via Ψ and its conjugate. By definition, we have

$$\frac{1}{\Omega} = \frac{1}{\Omega^2} + \frac{2}{\Omega} \frac{U^\dagger U}{\Omega} + \left| \frac{U^T U}{\Omega} \right|^2. \quad (\text{B.7})$$

This is equivalent to

$$\left(\frac{1}{\Omega} - \frac{1}{4} \Lambda \right)^2 = \frac{1}{16} \left(\Lambda^2 - |\Psi^T \Psi|^2 \right), \quad (\text{B.8})$$

where

$$\Lambda = 1 + \sqrt{1 - 2\Psi^\dagger \Psi + |\Psi^T \Psi|^2}. \quad (\text{B.9})$$

The consistency of eq. (B.8) can be seen to require

$$\Psi^\dagger \Psi < 1. \quad (\text{B.10})$$

Since for $\Psi \rightarrow 0$ we should have $\Omega \rightarrow 1$, it is necessary to choose the “plus” solution of (B.8), that is

$$\frac{4}{\Omega} = \Lambda + \sqrt{\Lambda^2 - |\Psi^T \Psi|^2} = \Lambda + \sqrt{2(\Lambda - \Psi^\dagger \Psi)}. \quad (\text{B.11})$$

The above consideration corresponds to the origin, $\Phi = 0$, of the base manifold. To extend these results to an arbitrary point Φ of the base manifold, we should replace

$$\Psi^\dagger \Psi \rightarrow g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j, \quad |\Psi^T \Psi|^2 \rightarrow 2(g^{\bar{i}j} \bar{\psi}_{\bar{i}} \psi_j)^2 - \frac{1}{2} R^{\bar{i}j\bar{k}l} \bar{\psi}_{\bar{i}} \psi_j \bar{\psi}_{\bar{k}} \psi_l. \quad (\text{B.12})$$

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